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Research Article

Characteristics of the Feasible Solution Set of Optimization Problems with Bipolar Fuzzy Relational Equality Constraints Defined by the Max–Einstein-Product Composition

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Abstract

Abstract: This paper studies the structure of feasible solution sets for optimization problems subject to bipolar fuzzy relational equality constraints defined by the max–Einstein-product composition, where the Einstein product is a continuous strict t-norm. We derive explicit one-dimensional feasible sets for the individual bipolar constraints and use them to obtain necessary and sufficient feasibility conditions for the complete system. The feasible solution set is represented as the union of finitely many compact sets; equivalently, it can be decomposed into a finite union of closed convex cells and need not be connected. An admissible-function representation is developed to describe the complete feasible region, and an algorithm is given to implement the resulting resolution procedure. A step-by-step numerical example illustrates the construction.
Keywords: Bipolar fuzzy relational equations, Einstein-product t-norm, max-t-norm composition, feasible solution set, optimization.
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1 Introduction

The theory of fuzzy relational equations (FREs) was introduced by Sanchez as a formal model for imprecise concepts represented through max–min composition [1]. Owing to the flexibility of fuzzy-set models, FREs have been used in a wide range of applications, including medical diagnosis, reasoning

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systems, data transmission, and engineering decision problems [1–4]. Consequently, a substantial literature has developed around solvability, solution-set characterization, and optimization under fuzzy relational constraints.

A basic task in an FRE problem is to determine whether the system is solvable and, when it is, to characterize its solution set. For FREs with continuous max-t-norm compositions, a nonempty solution set can be nonconvex and is often described through extremal solutions [5]. The computational difficulty can also be substantial: Chen and Wang [6,8] developed algorithms for minimal solutions, while Markovskii [7] related max-product FREs to the covering problem. Extensions to other t-norms and to fuzzy relational inequalities (FRIs) have subsequently been investigated in [9–20].

Optimization problems subject to FRE or FRI constraints form another active line of research [11,15–17,19,21–30]. One family of approaches converts the original model into an integer or mixed-integer programming problem; another analyzes the structure of the feasible region directly and then exploits that structure in the optimization stage. Representative contributions include the branch-and-bound approach of Fang and Li [33], the max-t-norm models in [24,26,28], and optimization methods for max-average and other compositions [29,36].

Bipolar fuzzy relational equations generalize ordinary FREs by incorporating positive and negative information simultaneously. Such models have been studied with max–min, max-product, max–Lukasiewicz, and more general max-strict compositions [27,38–45]. In particular, Ghodousian and Chopannavaz [45] considered bipolar FREs generated by max-strict t-norms, while a still broader framework based on continuous Archimedean t-norms was developed in [46].

In this paper we consider the optimization model

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & A^{+}\varphi x \vee A^{-}\varphi(\mathbf{1}-x) = b, \\ & x \in [0,1]^n, \end{array} \tag{1}$$

where $A^{+} = (a_{ij}^{+})_{m\times n}$ and $A^{-} = (a_{ij}^{-})_{m\times n}$ are fuzzy matrices, $b = (b_i)_{m\times 1}$ is a fuzzy vector, and

$$0 \le a_{ij}^{+}, a_{ij}^{-}, b_i \le 1,$$

for $i \in \mathcal{I} := \{1,\dots,m\}$ and $j \in \mathcal{J} := \{1,\dots,n\}$. The symbol $\vee$ denotes the maximum operation, $\mathbf{1}$ is the all-ones vector, and φ denotes the Einstein-product t-norm

$$T_{\mathrm{EP}}(x,y) = \frac{xy}{2-x-y+xy}, \qquad x,y \in [0,1]. \tag{2}$$

Accordingly, the ith bipolar constraint in (1) is

$$\max_{1\le j\le n} \Big\{ \max\{T_{\mathrm{EP}}(a_{ij}^{+}, x_j), T_{\mathrm{EP}}(a_{ij}^{-}, 1-x_j)\} \Big\} = b_i, \qquad i \in \mathcal{I}. \tag{3}$$

The Einstein product is increasing in each variable. For $a \ge b$ and $(a,b) \ne (0,0)$, the equation $T_{\mathrm{EP}}(a,x) = b$ has the unique solution

$$x = \frac{(2-a)b}{a+b-ab}.$$

This inverse relation is the basis of the interval formulas developed below.

The remainder of the paper is organized as follows. Section 2 gives the one-dimensional feasible sets associated with each coefficient pair. Section 3 derives feasibility criteria, an admissible-function representation, and an algorithm for the complete feasible set. Section 4 presents a numerical example.

2 Preliminaries and properties of the Einstein product

For each $i \in \mathcal{I}$, let S_i denote the feasible solution set of the ith equation in (3), namely

$$S_i = \left\{ x \in [0,1]^n : \max_{1 \le j \le n} \left\{ \max\{T_{\mathrm{EP}}(a_{ij}^+, x_j), T_{\mathrm{EP}}(a_{ij}^-, 1 - x_j)\} \right\} = b_i \right\}.$$

The feasible solution set of (1) is therefore

$$\mathcal{S}(A^+, A^-, b) = \bigcap_{i \in \mathcal{I}} S_i.$$

Definition 2.1. For $i \in \mathcal{I}$ and $j \in \mathcal{J}$, define

$$S_{ij}^+ = \{x \in [0,1] : T_{\mathrm{EP}}(a_{ij}^+, x) = b_i\}, \qquad S_{ij}^- = \{x \in [0,1] : T_{\mathrm{EP}}(a_{ij}^-, 1 - x) = b_i\},$$

and

$$I_{ij}^+ = \{x \in [0,1] : T_{\mathrm{EP}}(a_{ij}^+, x) \le b_i\}, \qquad I_{ij}^- = \{x \in [0,1] : T_{\mathrm{EP}}(a_{ij}^-, 1 - x) \le b_i\}.$$

For notational convenience, whenever $a_{ij}^\pm \ge b_i$, set

$$\theta_{ij}^\pm = \frac{(2 - a_{ij}^\pm) b_i}{a_{ij}^\pm + b_i - a_{ij}^\pm b_i},$$

when the denominator is nonzero. To include the degenerate case $a_{ij}^\pm = b_i = 0$, define the endpoints

$$\ell_{ij}^\pm = \begin{cases} 0, & b_i = 0, \\ \theta_{ij}^\pm, & b_i > 0, \end{cases} \qquad u_{ij}^\pm = \begin{cases} 1, & a_{ij}^\pm = b_i, \\ \theta_{ij}^\pm, & a_{ij}^\pm > b_i. \end{cases}.$$

Then the four one-dimensional sets are

$$S_{ij}^+ = \begin{cases} [\ell_{ij}^+, u_{ij}^+], & a_{ij}^+ \ge b_i, \\ \varnothing, & a_{ij}^+ < b_i, \end{cases} \qquad I_{ij}^+ = \begin{cases} [0, u_{ij}^+], & a_{ij}^+ \ge b_i, \\ [0,1], & a_{ij}^+ < b_i, \end{cases}, \tag{4a}$$

and

$$S_{ij}^- = \begin{cases} [1 - u_{ij}^-, 1 - \ell_{ij}^-], & a_{ij}^- \ge b_i, \\ \varnothing, & a_{ij}^- < b_i, \end{cases} \qquad I_{ij}^- = \begin{cases} [1 - u_{ij}^-, 1], & a_{ij}^- \ge b_i, \\ [0,1], & a_{ij}^- < b_i. \end{cases}. \tag{4b}$$

In the nondegenerate case these intervals reduce to singletons whenever equality is required.

Definition 2.2. For $i \in \mathcal{I}$ and $j \in \mathcal{J}$, define

$$S_{ij} = \left\{ x \in [0,1] : \max\{T_{\mathrm{EP}}(a_{ij}^{+}, x), T_{\mathrm{EP}}(a_{ij}^{-}, 1-x)\} = b_i \right\},$$

and

$$I_{ij} = \left\{ x \in [0,1] : \max\{T_{\mathrm{EP}}(a_{ij}^{+}, x), T_{\mathrm{EP}}(a_{ij}^{-}, 1-x)\} \le b_i \right\}.$$

Lemma 2.3. *For every $i \in \mathcal{I}$ and $j \in \mathcal{J}$,*

$$I_{ij} = I_{ij}^{+} \cap I_{ij}^{-},$$

and

$$S_{ij} = I_{ij} \cap \big(S_{ij}^{+} \cup S_{ij}^{-}\big).$$

Proof. The first identity follows directly from the definition of the maximum. For the second, $x \in S_{ij}$ if and only if both component values are at most b_i and at least one of them equals b_i. This is exactly the condition

$$x \in I_{ij} \cap \big(S_{ij}^{+} \cup S_{ij}^{-}\big).$$

■

Corollary 2.4. *Let $i \in \mathcal{I}$ and $j \in \mathcal{J}$.*

(a) If $a_{ij}^{+} < b_i$ and $a_{ij}^{-} < b_i$, then

$$I_{ij} = [0,1], \qquad S_{ij} = \varnothing.$$

(b) If $a_{ij}^{+} \ge b_i$ and $a_{ij}^{-} < b_i$, then

$$I_{ij} = [0, u_{ij}^{+}], \qquad S_{ij} = [\ell_{ij}^{+}, u_{ij}^{+}].$$

(c) If $a_{ij}^{-} \ge b_i$ and $a_{ij}^{+} < b_i$, then

$$I_{ij} = [1-u_{ij}^{-}, 1], \qquad S_{ij} = [1-u_{ij}^{-}, 1-\ell_{ij}^{-}].$$

(d) If $a_{ij}^{+} \ge b_i$ and $a_{ij}^{-} \ge b_i$, then

$$I_{ij} = [1-u_{ij}^{-}, u_{ij}^{+}],$$

and

$$S_{ij} = \begin{cases} [1-u_{ij}^{-}, 1-\ell_{ij}^{-}] \cup [\ell_{ij}^{+}, u_{ij}^{+}], & \ell_{ij}^{+} > 1-\ell_{ij}^{-}, \\ [1-u_{ij}^{-}, u_{ij}^{+}], & \ell_{ij}^{+} \le 1-\ell_{ij}^{-}. \end{cases}$$

Definition 2.5. For each $j \in \mathcal{J}$, define

$$\mathcal{I}^{-}(j) = \{i \in \mathcal{I} : a_{ij}^{-} \ge b_i\}, \qquad \mathcal{I}^{+}(j) = \{i \in \mathcal{I} : a_{ij}^{+} \ge b_i\},$$

and

$$I_j = \bigcap_{i \in \mathcal{I}} I_{ij}.$$

Also set

$$S'_{ij} = S_{ij} \cap I_j,$$

for $i \in \mathcal{I}$ and $j \in \mathcal{J}$.

Remark 2.6. For every $j \in \mathcal{J}$, $I_j = [L_j, U_j]$, where

$$L_j = \begin{cases} \max\limits_{i \in \mathcal{I}^-(j)} \{1 - u_{ij}^-\}, & \mathcal{I}^-(j) \neq \varnothing, \\ 0, & \mathcal{I}^-(j) = \varnothing, \end{cases},$$

and

$$U_j = \begin{cases} \min\limits_{i \in \mathcal{I}^+(j)} \{u_{ij}^+\}, & \mathcal{I}^+(j) \neq \varnothing, \\ 1, & \mathcal{I}^+(j) = \varnothing. \end{cases}.$$

3 Resolution of the feasible solution set

We first record two necessary feasibility conditions.

Lemma 3.1. *If $\mathcal{S}(A^+, A^-, b) \neq \varnothing$, then:*

(a) $I_j \neq \varnothing$ for every $j \in \mathcal{J}$;

(b) for every $i \in \mathcal{I}$, there exists at least one $j_i \in \mathcal{J}$ such that $S'_{ij_i} \neq \varnothing$.

Proof. Let $x \in \mathcal{S}(A^+, A^-, b)$. For every $i \in \mathcal{I}$ and $j \in \mathcal{J}$, feasibility of the ith equation implies

$$\max\{T_{\mathrm{EP}}(a_{ij}^+, x_j), T_{\mathrm{EP}}(a_{ij}^-, 1 - x_j)\} \leq b_i,$$

so $x_j \in I_{ij}$ for every i. Hence $x_j \in \bigcap_{i \in \mathcal{I}} I_{ij} = I_j$, proving (a).

For a fixed $i \in \mathcal{I}$, the maximum in the ith equation equals b_i, so there exists $j_i \in \mathcal{J}$ for which

$$\max\{T_{\mathrm{EP}}(a_{ij_i}^+, x_{j_i}), T_{\mathrm{EP}}(a_{ij_i}^-, 1 - x_{j_i})\} = b_i.$$

Thus $x_{j_i} \in S_{ij_i}$. Part (a) gives $x_{j_i} \in I_{j_i}$, and therefore $x_{j_i} \in S'_{ij_i}$. Hence $S'_{ij_i} \neq \varnothing$. ∎

Lemma 3.2. *A point $x \in [0,1]^n$ belongs to $\mathcal{S}(A^+, A^-, b)$ if and only if the following conditions hold:*

(i) $x_j \in I_j$ for every $j \in \mathcal{J}$;

(ii) for every $i \in \mathcal{I}$, there exists at least one $j_i \in \mathcal{J}$ such that $x_{j_i} \in S'_{ij_i}$.

Proof. Assume (i) and (ii). From (i), $x_j \in I_{ij}$ for every i and j, so

$$\max\{T_{\mathrm{EP}}(a_{ij}^+, x_j), T_{\mathrm{EP}}(a_{ij}^-, 1 - x_j)\} \leq b_i.$$

Condition (ii) guarantees, for each fixed i, an index j_i at which equality holds. Consequently,

$$\max_{1 \leq j \leq n} \Big\{ \max\{T_{\mathrm{EP}}(a_{ij}^+, x_j), T_{\mathrm{EP}}(a_{ij}^-, 1 - x_j)\} \Big\} = b_i,$$

for every $i \in \mathcal{I}$, and hence $x \in \mathcal{S}(A^+, A^-, b)$.

Conversely, if $x \in \mathcal{S}(A^+, A^-, b)$, then condition (i) follows exactly as in Lemma 3.1(a). For each i, the finite maximum in (3) must attain the value b_i at some coordinate j_i; therefore $x_{j_i} \in S_{ij_i} \cap I_{j_i} = S'_{ij_i}$, which gives (ii). ∎

Corollary 3.3. *For a fixed $i \in \mathcal{I}$, a point $x \in [0,1]^n$ belongs to S_i if and only if $x_j \in I_{ij}$ for every $j \in \mathcal{J}$ and there exists at least one $j_i \in \mathcal{J}$ such that $x_{j_i} \in S_{ij_i}$.*

Definition 3.4. For each $i \in \mathcal{I}$ and $j \in \mathcal{J}$, define

$$\mathcal{J}_i = \{j \in \mathcal{J} : S'_{ij} \neq \varnothing\}, \qquad \mathcal{I}_j = \{i \in \mathcal{I} : S'_{ij} \neq \varnothing\}.$$

Definition 3.5. A function $e : \mathcal{I} \to \mathcal{J}$ is called *admissible* if $e(i) \in \mathcal{J}_i(e)$ for every $i \in \mathcal{I}$, where

$$\mathcal{J}_1(e) = \mathcal{J}_1,$$

$$\mathcal{I}_j(e,i) = \{k \in \mathcal{I} : 1 \leq k < i \text{ and } e(k) = j\},$$

and, for $i \geq 2$,

$$\mathcal{J}_i(e) = \left\{ j \in \mathcal{J}_i : \ \mathcal{I}_j(e,i) = \varnothing \text{ or } S'_{ij} \cap \bigcap_{k \in \mathcal{I}_j(e,i)} S'_{kj} \neq \varnothing \right\}.$$

Let $\mathcal{E}$ denote the set of all admissible functions. We write an admissible function as the vector

$$e = [j_1, \ldots, j_m], \qquad e(i) = j_i.$$

Definition 3.6. For $e \in \mathcal{E}$, define

$$\mathcal{I}_j(e) = \{i \in \mathcal{I} : e(i) = j\},$$

and let $S(e)$ be the Cartesian product

$$S(e) = \prod_{j=1}^{n} S(e)_j, \qquad S(e)_j = \begin{cases} \bigcap_{i \in \mathcal{I}_j(e)} S'_{ij}, & \mathcal{I}_j(e) \neq \varnothing, \\ I_j, & \mathcal{I}_j(e) = \varnothing. \end{cases} \tag{5}$$

Corollary 3.7. *Let $e : \mathcal{I} \to \mathcal{J}$ satisfy $e(i) \in \mathcal{J}_i$ for every $i \in \mathcal{I}$. Then $e \in \mathcal{E}$ if and only if*

$$\bigcap_{i \in \mathcal{I}_j(e)} S'_{ij} \neq \varnothing,$$

for every $j \in \mathcal{J}$ such that $\mathcal{I}_j(e) \neq \varnothing$.

Proof. If $e \in \mathcal{E}$, the definition of admissibility ensures that, whenever another row is assigned to a previously used coordinate j, the new set S'_{ij} has nonempty intersection with the sets already assigned to that coordinate. Induction over i gives the stated nonempty intersection.

Conversely, assume all final intersections are nonempty. At every stage i, the previously used index set $\mathcal{I}_j(e,i)$ is contained in $\mathcal{I}_j(e)$. Hence, if $e(i) = j$ and $\mathcal{I}_j(e,i) \neq \varnothing$, then

$$S'_{ij} \cap \bigcap_{k \in \mathcal{I}_j(e,i)} S'_{kj} \neq \varnothing,$$

so $e(i) \in \mathcal{J}_i(e)$. Thus $e \in \mathcal{E}$. ∎

Remark 3.8. The number of admissible functions satisfies

$$|\mathcal{E}| \le \prod_{i\in\mathcal{I}} |\mathcal{J}_i|,$$

although the actual number is generally much smaller because incompatible coordinate assignments are discarded.

Theorem 3.9. *The feasible solution set of (1) is*

$$\mathcal{S}(A^+, A^-, b) = \bigcup_{e\in\mathcal{E}} S(e). \tag{6}$$

Proof. Let $x \in S(e_0)$ for some $e_0 \in \mathcal{E}$. By (5), $x_j \in I_j$ for every j. Moreover, for each $i \in \mathcal{I}$, if $j_i = e_0(i)$, then $i \in \mathcal{I}_{j_i}(e_0)$ and

$$x_{j_i} \in \bigcap_{k\in\mathcal{I}_{j_i}(e_0)} S'_{kj_i} \subseteq S'_{ij_i}.$$

Lemma 3.2 therefore yields $x \in \mathcal{S}(A^+, A^-, b)$.

Conversely, let $x \in \mathcal{S}(A^+, A^-, b)$ and define

$$\mathcal{J}_i(x) = \{j \in \mathcal{J} : x_j \in S'_{ij}\}.$$

By Lemma 3.2, $\mathcal{J}_i(x) \ne \varnothing$ for every i. Choose, for example,

$$e_0(i) = \min \mathcal{J}_i(x), \qquad i \in \mathcal{I}.$$

If several rows are assigned by e_0 to the same coordinate j, then x_j belongs to every corresponding S'_{ij}, so their intersection is nonempty. Corollary 3.4 shows that $e_0 \in \mathcal{E}$. In addition, $x_j \in I_j$ for every coordinate not used by e_0, while x_j lies in the appropriate intersection for every used coordinate. Thus $x \in S(e_0)$, proving (6). ∎

Corollary 3.10. *The feasible solution set $\mathcal{S}(A^+, A^-, b)$ is a finite union of compact sets. More precisely, each factor $S(e)_j$ is a finite union of closed intervals, so every $S(e)$ is a finite union of closed boxes. Consequently, $\mathcal{S}(A^+, A^-, b)$ can be represented as a finite union of closed convex cells and need not be connected.*

Proof. By Corollary 2.3, every S_{ij} is either empty, a closed interval, or the union of two closed intervals. Hence every $S'_{ij} = S_{ij} \cap I_j$ has the same finite-union property, and so does any finite intersection of such sets. Formula (5) therefore expresses $S(e)$ as a finite union of Cartesian products of closed intervals, each of which is a closed convex box. Since $\mathcal{E}$ is finite, (6) proves the assertion. ∎

The preceding results lead to the following resolution procedure.

Algorithm 1: Resolution of Problem (1)

Input: The matrices A^+ and A^- and the vector b.
Output: The feasible solution set $\mathcal{S}(A^+, A^-, b)$.

1. Compute I_{ij}, S_{ij}, I_j, and S'_{ij} for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$.
2. If $I_j = \varnothing$ for some $j \in \mathcal{J}$, or if $\mathcal{J}_i = \varnothing$ for some $i \in \mathcal{I}$, stop: the problem is infeasible.
3. Apply any available simplification rules to determine variables and remove redundant equations or columns.
4. Construct the admissible functions $e \in \mathcal{E}$ for the remaining system and compute the associated sets $S(e)$.
5. Restore any variables determined in Step 3 and generate
$$\mathcal{S}(A^+, A^-, b) = \bigcup_{e \in \mathcal{E}} S(e).$$

4 Numerical example

Consider Problem (1) with the Einstein-product t-norm and

$$A^+ = \begin{bmatrix} 0.84 & 0.36 & 0.01 & 0.43 & 0.25 & 0.51 \\ 0.98 & 0.12 & 0.42 & 0.16 & 0.94 & 0.01 \\ 0.75 & 0.76 & 0.75 & 0.46 & 0.50 & 0.80 \\ 0.80 & 0.72 & 0.39 & 0.55 & 0.90 & 0.52 \\ 0.13 & 0.10 & 0.02 & 0.46 & 0.64 & 0.61 \end{bmatrix},$$

$$A^- = \begin{bmatrix} 0.05 & 0.50 & 0.41 & 0.22 & 0.18 & 0.23 \\ 0.19 & 0.05 & 0.91 & 0.38 & 0.33 & 0.14 \\ 0.43 & 0.59 & 0.34 & 0.80 & 0.40 & 0.51 \\ 0.11 & 0.45 & 0.87 & 0.66 & 0.06 & 0.34 \\ 0.85 & 0.35 & 0.01 & 0.28 & 0.13 & 0.84 \end{bmatrix}, \qquad b = \begin{bmatrix} 0.36 \\ 0.56 \\ 0.76 \\ 0.66 \\ 0.46 \end{bmatrix}.$$

Here $\mathcal{I} = \{1, \ldots, 5\}$ and $\mathcal{J} = \{1, \ldots, 6\}$.

Step 1. Using Corollary 2.3, we obtain the sets I_{ij} and S_{ij} shown in Tables 1 and 2. Intersecting the I_{ij} columnwise gives the intervals I_j in Table 3, and then $S'_{ij} = S_{ij} \cap I_j$ gives Table 4. For example, for $i = 2$ and $j = 3$,

$$a^+_{23} = 0.42, \qquad a^-_{23} = 0.91, \qquad b_2 = 0.56,$$

so

$$I_{23} = [1 - u^-_{23}, 1] = [0.36, 1], \qquad S_{23} = \{0.36\}.$$

Since $I_3 = [0.36, 1]$, it follows that

$$S'_{23} = S_{23} \cap I_3 = \{0.36\}.$$

Step 2. Table 3 shows that $I_j \neq \varnothing$ for every $j \in \mathcal{J}$.

Table 1: Sets I_{ij} for $i \in \mathcal{I}$ and $j \in \mathcal{J}$.

$[0, 0.47]$	$[0.21, 1]$	$[0.08, 1]$	$[0, 0.89]$	$[0, 1]$	$[0, 0.78]$
$[0, 0.58]$	$[0, 1]$	$[0.36, 1]$	$[0, 1]$	$[0, 0.61]$	$[0, 1]$
$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0.04, 1]$	$[0, 1]$	$[0, 0.96]$
$[0, 0.85]$	$[0, 0.93]$	$[0.22, 1]$	$[0, 1]$	$[0, 0.75]$	$[0, 1]$
$[0.42, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 0.78]$	$[0.42, 0.81]$

Table 2: Sets S_{ij} for $i \in \mathcal{I}$ and $j \in \mathcal{J}$.

$\{0.47\}$	$\{0.21, 1\}$	$\{0.08\}$	$\{0.89\}$	$\varnothing$	$\{0.78\}$
$\{0.58\}$	$\varnothing$	$\{0.36\}$	$\varnothing$	$\{0.61\}$	$\varnothing$
$\varnothing$	$\{1\}$	$\varnothing$	$\{0.04\}$	$\varnothing$	$\{0.96\}$
$\{0.85\}$	$\{0.93\}$	$\{0.22\}$	$\{0\}$	$\{0.75\}$	$\varnothing$
$\{0.42\}$	$\varnothing$	$\varnothing$	$\{1\}$	$\{0.78\}$	$\{0.42, 0.81\}$

Step 3. From Table 4,

$$\mathcal{J}_1 = \{1, 2, 4, 6\}, \quad \mathcal{J}_2 = \{3, 5\}, \quad \mathcal{J}_3 = \{4\}, \quad \mathcal{J}_4 = \{2\}, \quad \mathcal{J}_5 = \{1, 6\}.$$

Hence

$$|\mathcal{E}| \leq \prod_{i \in \mathcal{I}} |\mathcal{J}_i| = 4 \times 2 \times 1 \times 1 \times 2 = 16.$$

For illustration, the function

$$e_1 = [1, 3, 4, 2, 1],$$

is not admissible because rows 1 and 5 are both assigned to coordinate 1, whereas

$$S'_{11} \cap S'_{51} = \{0.47\} \cap \{0.42\} = \varnothing.$$

On the other hand,

$$e_2 = [1, 3, 4, 2, 6],$$

is admissible and gives

$$S(e_2) = \{0.47\} \times \{0.93\} \times \{0.36\} \times \{0.04\} \times [0, 0.61] \times \{0.42\}.$$

Step 4. Applying the simplification strategy in [30], the second row is removed and the fifth variable is fixed at

$$x_5 = 0.61.$$

After deleting the corresponding fifth column, the reduced matrices consist of four rows (original rows $1, 3, 4, 5$) and five columns (original columns $1, 2, 3, 4, 6$):

$$\widehat{A}^+ = \begin{bmatrix} 0.84 & 0.36 & 0.01 & 0.43 & 0.51 \\ 0.75 & 0.76 & 0.75 & 0.46 & 0.80 \\ 0.80 & 0.72 & 0.39 & 0.55 & 0.52 \\ 0.13 & 0.10 & 0.02 & 0.46 & 0.61 \end{bmatrix},$$

Table 3: Sets $I_j = [L_j, U_j]$ for $j \in \mathcal{J}$.

$[0.42, 0.47]$	$[0.21, 0.93]$	$[0.36, 1]$	$[0.04, 0.89]$	$[0, 0.61]$	$[0.42, 0.78]$

Table 4: Sets S'_{ij} for $i \in \mathcal{I}$ and $j \in \mathcal{J}$.

$\{0.47\}$	$\{0.21\}$	$\varnothing$	$\{0.89\}$	$\varnothing$	$\{0.78\}$
$\varnothing$	$\varnothing$	$\{0.36\}$	$\varnothing$	$\{0.61\}$	$\varnothing$
$\varnothing$	$\varnothing$	$\varnothing$	$\{0.04\}$	$\varnothing$	$\varnothing$
$\varnothing$	$\{0.93\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$\{0.42\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{0.42\}$

$$\widehat{A}^- = \begin{bmatrix} 0.05 & 0.50 & 0.41 & 0.22 & 0.23 \\ 0.43 & 0.59 & 0.34 & 0.80 & 0.51 \\ 0.11 & 0.45 & 0.87 & 0.66 & 0.34 \\ 0.85 & 0.35 & 0.01 & 0.28 & 0.84 \end{bmatrix}, \qquad \widehat{b} = \begin{bmatrix} 0.36 \\ 0.76 \\ 0.66 \\ 0.46 \end{bmatrix}.$$

The corresponding nonempty S'_{ij} sets are displayed in Table 5, with columns indexed locally by the five retained variables.

Table 5: Reduced sets S'_{ij} after the simplification step.

$\{0.47\}$	$\{0.21\}$	$\varnothing$	$\{0.89\}$	$\{0.78\}$
$\varnothing$	$\varnothing$	$\varnothing$	$\{0.04\}$	$\varnothing$
$\varnothing$	$\{0.93\}$	$\varnothing$	$\varnothing$	$\varnothing$
$\{0.42\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{0.42\}$

Steps 5 and 6. For the reduced system, the eight candidate assignments are

$$\begin{aligned} e_1 &= [1,4,2,1], \quad e_2 = [1,4,2,5], \quad e_3 = [2,4,2,1], \quad e_4 = [2,4,2,5], \\ e_5 &= [4,4,2,1], \quad e_6 = [4,4,2,5], \quad e_7 = [5,4,2,1], \quad e_8 = [5,4,2,5]. \end{aligned}$$

Checking Definition 3.6 shows that only e_2 and e_7 are admissible. Restoring $x_5 = 0.61$, we obtain

$$S(e_2) = \{0.47\} \times \{0.93\} \times [0.36, 1] \times \{0.04\} \times \{0.61\} \times \{0.42\},$$

and

$$S(e_7) = \{0.42\} \times \{0.93\} \times [0.36, 1] \times \{0.04\} \times \{0.61\} \times \{0.78\}.$$

Therefore,

$$\mathcal{S}(A^+, A^-, b) = S(e_2) \cup S(e_7).$$

This explicit description also shows that the feasible set can be disconnected.

5 Conclusion

We have characterized the feasible solution set of optimization problems subject to bipolar fuzzy relational equalities defined by the max–Einstein-product composition. Explicit one-dimensional feasible sets lead to necessary and sufficient conditions for feasibility and to an admissible-function representation of the complete solution set. The resulting feasible region is a finite union of compact sets and, more specifically, a finite union of closed convex cells; it need not be connected. The accompanying resolution algorithm summarizes the construction, and the numerical example demonstrates how the admissible sets and simplification steps are combined in practice.

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