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*Research Article*

# Fixed Point Theorems for $(\alpha_s, \mu_s, (Q, h))$-Interpolative Contractions in $S$-Metric Spaces

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**Abstract**

**Abstract:** We introduce $(\alpha_s, \mu_s, (Q, h))$-interpolative Kannan- and Čirić–Reich–Rus-type contractions in $S$-metric spaces. The framework combines the admissible control function $\alpha_s$, the subadmissible control function $\mu_s$, and a pair of upper-class functions $(Q, h)$. Fixed point theorems are established under continuity and, alternatively, under weaker sequential assumptions. Suitable choices of $(Q, h)$ yield several power- and exponential-type consequences and clarify the connection with familiar Banach-type contractive conditions. Two examples illustrate the results; one of them has two distinct fixed points, showing that the existence theorems do not in general imply uniqueness.

**Keywords:** Fixed point, $\alpha$-admissible mapping, orbital admissibility, $S$-metric space, interpolative contraction, upper-class functions.
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## 1 Introduction

The Banach contraction principle [1] is a cornerstone of metric fixed point theory. The search for weaker sufficient conditions guaranteeing the existence of fixed points has led to many variants of the classical contraction condition. Kannan [2,3], for example, replaced the distance between two points by distances from the points to their own images. Reich [4–6], Rus [7,8], and Čirić [9–11]

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subsequently developed related contractive conditions combining terms such as $d(x,y)$, $d(x,Tx)$, and $d(y,Ty)$. A representative inequality is

$$d(Tx,Ty) \leq \lambda\big[d(x,y) + d(x,Tx) + d(y,Ty)\big], \qquad \lambda \in \left[0, \frac{1}{3}\right),$$

for all $x, y \in X$, while further versions allow the three terms to have independent positive coefficients whose sum is less than one.

A second direction, initiated by Karapınar [12], is based on interpolative contractions. A mapping $T$ on a metric space $(X,d)$ is called an interpolative Kannan-type contraction if there exist $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \leq \lambda[d(x,Tx)]^{\alpha}[d(y,Ty)]^{1-\alpha},$$

for the prescribed nonfixed points. Karapınar, Agarwal, and Aydi [13] emphasized that such hypotheses need not force uniqueness and refined the domain on which the contractive estimate is imposed. Aydi, Karapınar, and Roldán López de Hierro [14] incorporated a comparison function $\psi$ from the class

$$\Psi = \left\{\psi : [0,\infty) \to [0,\infty) : \ \psi \text{ is nondecreasing and } \sum_{n=1}^{\infty} \psi^n(t) < \infty \text{ for every } t > 0\right\},$$

where $\psi^n$ denotes the $n$th iterate of $\psi$. Every $\psi \in \Psi$ satisfies $\psi(0) = 0$, $\psi(t) < t$ for $t > 0$, and $\psi^n(t) \to 0$ as $n \to \infty$.

The notion of an $\alpha$-admissible mapping was introduced by Samet, Vetro, and Vetro [15].

**Definition 1.1** ([15])**.** Let $T : X \to X$ and let $\alpha : X \times X \to [0,\infty)$. The mapping $T$ is called *$\alpha$-admissible* if

$$\alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx,Ty) \geq 1,$$

for all $x, y \in X$.

Popescu [16] introduced the corresponding orbital notion.

**Definition 1.2** ([16])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. The mapping $T$ is called *$\alpha$-orbital admissible* if

$$\alpha(x,Tx) \geq 1 \quad \Longrightarrow \quad \alpha(Tx,T^2x) \geq 1,$$

for every $x \in X$.

We next recall the concept of an $S$-metric space introduced by Sedghi, Shobe, and Aliouche [17].

**Definition 1.3** ([17])**.** Let $X$ be a nonempty set. A mapping $S : X^3 \to [0,\infty)$ is called an *$S$-metric* if, for all $x_1, x_2, x_3, t \in X$,

1. $S(x_1,x_2,x_3) \geq 0$;

2. $S(x_1,x_2,x_3) = 0$ if and only if $x_1 = x_2 = x_3$;

3. $$S(x_1,x_2,x_3) \leq S(x_1,x_1,t) + S(x_2,x_2,t) + S(x_3,x_3,t).$$

The pair $(X, S)$ is then called an $S$-metric space.

A standard consequence of the $S$-metric axioms is

$$S(x, x, y) = S(y, y, x),$$

for all $x, y \in X$. A sequence $\{x_n\}$ converges to $x$ if $S(x_n, x_n, x) \to 0$, and it is Cauchy if $S(x_n, x_n, x_m) \to 0$ as $m, n \to \infty$.

Priyobarta et al. [18] transferred admissibility ideas to $S$-metric spaces. The related orbital form used below appears in [21].

**Definition 1.4** ([18])**.** Let $(X, S)$ be an $S$-metric space, $T : X \to X$, and $\alpha_s : X^3 \to [0, \infty)$. The mapping $T$ is called *$\alpha_s$-admissible* if

$$\alpha_s(x, y, z) \geq 1 \quad \Longrightarrow \quad \alpha_s(Tx, Ty, Tz) \geq 1,$$

for all $x, y, z \in X$.

**Definition 1.5** ([21])**.** Let $(X, S)$ be an $S$-metric space, $T : X \to X$, and $\alpha_s : X^3 \to [0, \infty)$. The mapping $T$ is called *$\alpha_s$-orbital admissible* if

$$\alpha_s(x, x, Tx) \geq 1 \quad \Longrightarrow \quad \alpha_s(Tx, Tx, T^2x) \geq 1,$$

for every $x \in X$.

**Definition 1.6** ([21])**.** A mapping $T$ on an $S$-metric space $(X, S)$ is called an *$\alpha_s$-interpolative Kannan-type contraction* if there exist $\psi \in \Psi$, a mapping $\alpha_s : X^3 \to [0, \infty)$, and positive numbers $p, q$ with $p + q < 1$ such that

$$\alpha_s(x, y, z)S(Tx, Ty, Tz) \leq \psi\big([S(x, x, Tx)]^p[S(y, y, Ty)]^q[S(z, z, Tz)]^{1-p-q}\big)\,,$$

for all $x, y, z \in X \setminus \mathrm{Fix}(T)$.

The corresponding fixed point results from [21] may be stated as follows.

**Theorem 1.7** ([21])**.** *Let $T : X \to X$ be a mapping on a complete $S$-metric space $(X, S)$. Suppose that $T$ is continuous and $\alpha_s$-orbital admissible, that there exists $x_0 \in X$ with $\alpha_s(x_0, x_0, Tx_0) \geq 1$, and that $T$ is an $\alpha_s$-interpolative Kannan-type contraction. Then $T$ has a fixed point in $X$.*

We shall also use the following sequential condition:

If $\{x_n\}$ is a sequence in $X$ such that $\alpha_s(x_n, x_n, x_{n+1}) \geq 1$ for every $n$ and $x_n \to x \in X$, then there exists a subsequence $\{x_{n(k)}\}$ such that $\alpha_s(x_{n(k)}, x_{n(k)}, x) \geq 1$ for every $k$. (H)

**Theorem 1.8** ([21])**.** *Let $T : X \to X$ be a mapping on a complete $S$-metric space $(X, S)$. Suppose that condition (H) holds, $T$ is $\alpha_s$-orbital admissible, there exists $x_0 \in X$ with $\alpha_s(x_0, x_0, Tx_0) \geq 1$, and $T$ is an $\alpha_s$-interpolative Kannan-type contraction. Then $T$ has a fixed point in $X$.*

**Definition 1.9** ([21])**.** A mapping $T$ on an $S$-metric space $(X, S)$ is called an *$\alpha_s$-interpolative Ćirić–Reich–Rus-type contraction* if there exist $\psi \in \Psi$, a mapping $\alpha_s : X^3 \to [0, \infty)$, and positive numbers $p, q, r$ with $p + q + r < 1$ such that

$$\begin{aligned}\alpha_s(x, y, z)S(Tx, Ty, Tz) \leq \psi\big(&[S(x, y, z)]^p[S(x, x, Tx)]^q[S(y, y, Ty)]^r \\ &\times [S(z, z, Tz)]^{1-p-q-r}\big),\end{aligned}$$

for all $x, y, z \in X \setminus \mathrm{Fix}(T)$.

**Theorem 1.10** ([21])**.** *Let $T : X \to X$ be a mapping on a complete $S$-metric space $(X, S)$. Suppose that $T$ is continuous and $\alpha_s$-orbital admissible, that there exists $x_0 \in X$ with $\alpha_s(x_0, x_0, Tx_0) \geq 1$, and that $T$ is an $\alpha_s$-interpolative Ćirić–Reich–Rus-type contraction. Then $T$ has a fixed point in $X$.*

**Theorem 1.11** ([21])**.** *Under the assumptions of the preceding theorem, continuity of $T$ may be replaced by condition $(H)$. The resulting mapping still has a fixed point in $X$.*

The second control function used in this paper is the following subadmissible function.

**Definition 1.12** ([22])**.** Let $T : X \to X$ and let $\mu_s : X^3 \to [0, \infty)$. The mapping $T$ is called *$\mu_s$-subadmissible* if

$$\mu_s(x, y, z) \leq 1 \quad \Longrightarrow \quad \mu_s(Tx, Ty, Tz) \leq 1,$$

for all $x, y, z \in X$.

**Definition 1.13.** Let $(X, S)$ be an $S$-metric space, $T : X \to X$, and $\mu_s : X^3 \to [0, \infty)$. The mapping $T$ is called *$\mu_s$-orbital subadmissible* if

$$\mu_s(x, x, Tx) \leq 1 \quad \Longrightarrow \quad \mu_s(Tx, Tx, T^2x) \leq 1,$$

for every $x \in X$.

We finally recall the upper-class function framework from [23–25]. Throughout, $\mathbb{R}_+ = [0, \infty)$.

**Definition 1.14** ([23–25])**.** A function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is said to belong to a *subclass of type I* if

$$x \geq 1 \quad \Longrightarrow \quad h(1, y) \leq h(x, y),$$

for all $x, y \in \mathbb{R}_+$.

**Example 1.15** ([23–25])**.** The following are standard examples of functions in a subclass of type I:

1. $h(x, y) = (y + \ell)^x$, where $\ell > 1$;
2. $h(x, y) = (x + \ell)^y$, where $\ell > 0$;
3. $h(x, y) = x^n y^k$, where $k > 0$ and $n \in \mathbb{N} \cup \{0\}$;
4. $$h(x, y) = \frac{x^n + x^{n-1} + \cdots + x + 1}{n + 1}\, y.$$
5. $$h(x, y) = \left( \frac{x^n + x^{n-1} + \cdots + x + 1}{n + 1} + \ell \right)^y, \qquad \ell > 1.$$
6. $$h(x, y) = \frac{mx + n}{m + n}\, y, \qquad m, n \in \mathbb{N}.$$

**Remark 1.16.** Definition 1.14 requires only the monotonicity relation $h(1, y) \leq h(x, y)$ for $x \geq 1$. No additional inequality between $y$ and $h(1, y)$ is needed in the arguments below.

**Definition 1.17** ([23–25]). A function $h : \mathbb{R}_+^3 \to \mathbb{R}$ is said to belong to a *subclass of type II* if

$$x, y \geq 1 \quad \Longrightarrow \quad h(1,1,z) \leq h(x,y,z),$$

for all $x, y, z \in \mathbb{R}_+$.

**Example 1.18** ([23–25]). Examples of functions in a subclass of type II include:

1. $h(x,y,z) = (z+\ell)^{xy}$, where $\ell > 1$;
2. $h(x,y,z) = (xy+\ell)^z$, where $\ell > 0$;
3. $h(x,y,z) = x^m y^n z^p$, where $m, n \in \mathbb{N} \cup \{0\}$ and $p > 0$;
4. $$h(x,y,z) = \left( \frac{\sum_{i=0}^{n} x^{n-i} y^i}{n+1} \right) z.$$
5. $$h(x,y,z) = \left( \frac{\sum_{i=0}^{n} x^{n-i} y^i}{n+1} + \ell \right)^z, \qquad \ell > 1.$$
6. $$h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k, \qquad m, n, p, q, k \in \mathbb{N}.$$

**Remark 1.19.** As in the type-I case, no separate inequality between $z$ and $h(1,1,z)$ is required beyond the defining upper-class implication introduced next.

**Definition 1.20** ([23–25]). Let $h, Q : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$. The pair $(Q, h)$ is called an *upper class of type I* if $h$ is of subclass type I and

1. $$0 \leq s \leq 1 \quad \Longrightarrow \quad Q(s,t) \leq Q(1,t).$$
2. $$h(1,y) \leq Q(s,t) \quad \Longrightarrow \quad y \leq st,$$

   for all $y, s, t \in \mathbb{R}_+$. Setting $s = 1$ in the second implication gives the associated special upper-class condition.

**Example 1.21** ([23–25]). Typical upper-class pairs of type I are:

1. $h(x,y) = (y+\ell)^x$, $Q(s,t) = st + \ell$, $\ell > 1$;
2. $h(x,y) = (x+\ell)^y$, $Q(s,t) = (1+\ell)^{st}$, $\ell > 0$;
3. $h(x,y) = x^n y^k$, $Q(s,t) = s^m t^k$, with the parameters chosen so that the upper-class implication holds;
4. $h(x,y) = \dfrac{mx+n}{m+n} y$, $Q(s,t) = st$, $m, n \in \mathbb{N}$;

5.
$$h(x,y) = \left(\frac{x^n + x^{n-1} + \cdots + x + 1}{n+1} + \ell\right)^y, \qquad Q(s,t) = (1+\ell)^{st},$$
where $\ell > 1$;

6. $h(x,y) = (y+\ell)^x$, $Q(s,t) = st + \ell/k$, where $\ell > 1$ and $k \geq 1$.

**Definition 1.22** ([23–25])**.** Let $h : \mathbb{R}_+^3 \to \mathbb{R}$ and $Q : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$. The pair $(Q,h)$ is called an *upper class of type II* if $h$ is of subclass type II and

1.
$$0 \leq s \leq 1 \quad \Longrightarrow \quad Q(s,t) \leq Q(1,t).$$

2.
$$h(1,1,z) \leq Q(s,t) \quad \Longrightarrow \quad z \leq st,$$
for all $z, s, t \in \mathbb{R}_+$. Again, $s = 1$ gives the special upper-class condition.

**Example 1.23** ([23–25])**.** Examples of upper-class pairs of type II include:

1. $h(x,y,z) = (z+\ell)^{xy}$, $Q(s,t) = st + \ell$, $\ell > 1$;

2. $h(x,y,z) = (xy+\ell)^z$, $Q(s,t) = (1+\ell)^{st}$, $\ell > 0$;

3. $h(x,y,z) = x^m y^n z^p$, $Q(s,t) = s^p t^p$, $m, n \in \mathbb{N} \cup \{0\}$, $p > 0$;

4.
$$h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3}\, z^k, \qquad Q(s,t) = (st)^k,$$
where $m, n, p, q, k \in \mathbb{N}$;

5.
$$h(x,y,z) = \left(\frac{\sum_{i=0}^n x^{n-i} y^i}{n+1} + \ell\right)^z, \qquad Q(s,t) = (1+\ell)^{st},$$
where $\ell > 1$;

6.
$$h(x,y,z) = \left(\frac{\sum_{i=0}^n x^{n-i} y^i}{n+1}\right) z, \qquad Q(s,t) = st.$$

7. $h(x,y,z) = (z+\ell)^{xy}$, $Q(s,t) = st + \ell/k$, $\ell > 1$, $k \geq 1$.

These notions motivate the two classes of interpolative contractions introduced in the next section.

## 2 Main Results

We begin with the Kannan-type form of the new contractive condition.

**Definition 2.1.** Let $(X, S)$ be an $S$-metric space, let $(Q, h)$ be an upper-class pair of type I, and let $T : X \to X$. The mapping $T$ is called an $(\alpha_s, \mu_s, (Q, h))$-*interpolative Kannan-type contraction* if there exist $\psi \in \Psi$, functions $\alpha_s, \mu_s : X^3 \to [0, \infty)$, and positive numbers $p, q$ with $p + q < 1$ such that

$$h\big(\alpha_s(x, y, z), S(Tx, Ty, Tz)\big) \le Q\Big(\mu_s(x, y, z), \psi\big([S(x, x, Tx)]^p [S(y, y, Ty)]^q \times [S(z, z, Tz)]^{1-p-q}\big)\Big), \tag{1}$$

for all $x, y, z \in X \setminus \mathrm{Fix}(T)$.

**Theorem 2.2.** *Let $T : X \to X$ be a mapping on a complete $S$-metric space $(X, S)$. Assume that:*

1. *$T$ is continuous;*
2. *$T$ is $\alpha_s$-orbital admissible and $\mu_s$-orbital subadmissible;*
3. *there exists $x_0 \in X$ such that*
$$\alpha_s(x_0, x_0, Tx_0) \ge 1 \qquad \textit{and} \qquad \mu_s(x_0, x_0, Tx_0) \le 1.$$
4. *$T$ is an $(\alpha_s, \mu_s, (Q, h))$-interpolative Kannan-type contraction.*

*Then $T$ has a fixed point in $X$.*

*Proof.* Choose $x_0$ as in (3) and define

$$x_{n+1} = Tx_n \qquad (n \ge 0).$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0$, then $x_{n_0}$ is a fixed point. Hence assume that $x_n \ne x_{n+1}$ for all $n$. By orbital admissibility and subadmissibility,

$$\alpha_s(x_n, x_n, x_{n+1}) \ge 1 \qquad \text{and} \qquad \mu_s(x_n, x_n, x_{n+1}) \le 1, \qquad n \ge 0.$$

Set

$$d_n = S(x_n, x_n, x_{n+1}), \qquad n \ge 0.$$

Taking $x = y = x_{n-1}$ and $z = x_n$ in (1), and using $Tx_{n-1} = x_n$ and $Tx_n = x_{n+1}$, gives

$$\begin{aligned} h(1, d_n) &\le h\big(\alpha_s(x_{n-1}, x_{n-1}, x_n), d_n\big) \\ &\le Q\Big(\mu_s(x_{n-1}, x_{n-1}, x_n), \psi\big(d_{n-1}^{p+q} d_n^{1-p-q}\big)\Big) \\ &\le Q\Big(1, \psi\big(d_{n-1}^{p+q} d_n^{1-p-q}\big)\Big). \end{aligned}$$

By the special upper-class implication,

$$d_n \le \psi\big(d_{n-1}^{p+q} d_n^{1-p-q}\big). \tag{2}$$

Since $\psi(t) < t$ for $t > 0$, we obtain

$$d_n < d_{n-1}^{p+q} d_n^{1-p-q},$$

and hence

$$d_n < d_{n-1} \qquad (n \geq 1). \tag{3}$$

Thus $\{d_n\}$ is decreasing. Using (3) in (2) and the monotonicity of $\psi$, we get

$$d_n \leq \psi(d_{n-1}) \leq \cdots \leq \psi^n(d_0),$$

so $d_n \to 0$. Moreover, for $m > n$, repeated use of the $S$-metric triangle inequality yields

$$S(x_n, x_n, x_m) \leq 2 \sum_{i=n}^{m-1} d_i \leq 2 \sum_{i=n}^{\infty} \psi^i(d_0),$$

and the last expression tends to zero because $\psi \in \Psi$. Therefore $\{x_n\}$ is Cauchy. Completeness gives an $x \in X$ such that

$$S(x_n, x_n, x) \longrightarrow 0. \tag{4}$$

Since $T$ is continuous, $x_{n+1} = Tx_n \to Tx$, while $x_{n+1} \to x$. Hence $Tx = x$. ■

To remove the continuity assumption, we use the following counterpart of condition $(H)$:

If $\{x_n\}$ is a sequence in $X$ such that $\mu_s(x_n, x_n, x_{n+1}) \leq 1$ for every $n$ and $x_n \to x \in X$, then there exists a subsequence $\{x_{n(k)}\}$ such that $\mu_s(x_{n(k)}, x_{n(k)}, x) \leq 1$ for every $k$. $(H')$

**Theorem 2.3.** *Let $T : X \to X$ be a self-mapping on a complete $S$-metric space $(X, S)$. Assume that conditions $(H)$ and $(H')$ hold and that:*

1. *$T$ is $\alpha_s$-orbital admissible and $\mu_s$-orbital subadmissible;*
2. *there exists $x_0 \in X$ such that*

$$\alpha_s(x_0, x_0, Tx_0) \geq 1 \qquad \text{and} \qquad \mu_s(x_0, x_0, Tx_0) \leq 1.$$

3. *$T$ is an $(\alpha_s, \mu_s, (Q, h))$-interpolative Kannan-type contraction.*

*Then $T$ has a fixed point in $X$.*

*Proof.* Construct $\{x_n\}$ as in Theorem 2.2. The same argument shows that $\{x_n\}$ is Cauchy and converges to some $x \in X$, with (4) holding. If some $x_n$ is a fixed point, there is nothing to prove, so assume otherwise. Suppose, for a contradiction, that $x \neq Tx$.

By $(H)$ and $(H')$, after passing to a common subsequence if necessary, we may assume that

$$\alpha_s(x_{n(k)}, x_{n(k)}, x) \geq 1 \qquad \text{and} \qquad \mu_s(x_{n(k)}, x_{n(k)}, x) \leq 1,$$

for every $k$. Put $D = S(x, x, Tx) > 0$. Since $x_{n(k)} \to x$ and $d_{n(k)} = S(x_{n(k)}, x_{n(k)}, Tx_{n(k)}) \to 0$, for all sufficiently large $k$,

$$S(x_{n(k)}, x_{n(k)}, Tx_{n(k)}) \leq D.$$

Taking $x = y = x_{n(k)}$ and $z = x$ in (1), the upper-class properties give

$$S(x_{n(k)+1}, x_{n(k)+1}, Tx) \leq \psi\Big(d_{n(k)}^{p+q} D^{1-p-q}\Big) \leq \psi(D).$$

Letting $k \to \infty$ yields

$$D \leq \psi(D) < D,$$

a contradiction. Therefore $Tx = x$. ■

We next introduce the Čirić–Reich–Rus-type version.

**Definition 2.4.** Let $(X, S)$ be an $S$-metric space, let $(Q, h)$ be an upper-class pair of type I, and let $T : X \to X$. The mapping $T$ is called an $(\alpha_s, \mu_s, (Q, h))$-*interpolative Čirić–Reich–Rus-type contraction* if there exist $\psi \in \Psi$, functions $\alpha_s, \mu_s : X^3 \to [0, \infty)$, and positive numbers $p, q, r$ with $p + q + r < 1$ such that

$$\begin{aligned} h\Big(\alpha_s(x,y,z), S(Tx,Ty,Tz)\Big) \leq Q\Big(\mu_s(x,y,z), \psi\big([S(x,y,z)]^p [S(x,x,Tx)]^q \\ \times [S(y,y,Ty)]^r [S(z,z,Tz)]^{1-p-q-r}\big)\Big), \end{aligned} \tag{5}$$

for all $x, y, z \in X \setminus \mathrm{Fix}(T)$.

**Theorem 2.5.** *Let $T : X \to X$ be a self-mapping on a complete $S$-metric space $(X, S)$. Assume that:*

1. *$T$ is continuous;*
2. *$T$ is $\alpha_s$-orbital admissible and $\mu_s$-orbital subadmissible;*
3. *there exists $x_0 \in X$ such that*
$$\alpha_s(x_0, x_0, Tx_0) \geq 1 \qquad \text{and} \qquad \mu_s(x_0, x_0, Tx_0) \leq 1.$$
4. *$T$ is an $(\alpha_s, \mu_s, (Q, h))$-interpolative Čirić–Reich–Rus-type contraction.*

*Then $T$ has a fixed point in $X$.*

*Proof.* Choose $x_0$ as in (3), set $x_{n+1} = Tx_n$, and assume that no iterate is already fixed. Orbital admissibility and subadmissibility give

$$\alpha_s(x_n, x_n, x_{n+1}) \geq 1 \qquad \text{and} \qquad \mu_s(x_n, x_n, x_{n+1}) \leq 1, \qquad n \geq 0.$$

Again write $d_n = S(x_n, x_n, x_{n+1})$. Taking $x = y = x_{n-1}$ and $z = x_n$ in (5), and using the upper-class properties, yields

$$d_n \leq \psi\big(d_{n-1}^{p+q+r} d_n^{1-p-q-r}\big). \tag{6}$$

Consequently,

$$d_n < d_{n-1} \qquad (n \geq 1), \tag{7}$$

and hence

$$d_n \leq \psi(d_{n-1}) \leq \cdots \leq \psi^n(d_0) \longrightarrow 0. \tag{8}$$

As in the proof of Theorem 2.2, the summability of the iterates of $\psi$ shows that $\{x_n\}$ is Cauchy. Thus $x_n \to x$ for some $x \in X$, and continuity of $T$ gives $Tx = x$. ■

**Theorem 2.6.** *Let $T : X \to X$ be a self-mapping on a complete $S$-metric space $(X, S)$. Assume that conditions $(H)$ and $(H')$ hold and that:*

1. *$T$ is $\alpha_s$-orbital admissible and $\mu_s$-orbital subadmissible;*
2. *there exists $x_0 \in X$ such that*

$$\alpha_s(x_0, x_0, Tx_0) \geq 1 \qquad \textit{and} \qquad \mu_s(x_0, x_0, Tx_0) \leq 1.$$

3. *$T$ is an $(\alpha_s, \mu_s, (Q, h))$-interpolative Čirić–Reich–Rus-type contraction.*

*Then $T$ has a fixed point in $X$.*

*Proof.* By Theorem 2.5 up to the continuity step, the Picard sequence $\{x_n\}$ is Cauchy, $x_n \to x$ for some $x \in X$, and (8) holds. Assume that $x \neq Tx$ and put $D = S(x, x, Tx) > 0$. Conditions $(H)$ and $(H')$ provide a subsequence $\{x_{n(k)}\}$ satisfying

$$\alpha_s(x_{n(k)}, x_{n(k)}, x) \geq 1 \qquad \text{and} \qquad \mu_s(x_{n(k)}, x_{n(k)}, x) \leq 1,$$

for every $k$. For all sufficiently large $k$,

$$S(x_{n(k)}, x_{n(k)}, x) \leq D \qquad \text{and} \qquad S(x_{n(k)}, x_{n(k)}, Tx_{n(k)}) \leq D.$$

Taking $x = y = x_{n(k)}$ and $z = x$ in (5) and using the monotonicity of $\psi$ gives

$$S(x_{n(k)+1}, x_{n(k)+1}, Tx) \leq \psi(D).$$

Passing to the limit gives $D \leq \psi(D) < D$, a contradiction. Hence $Tx = x$. ■

## 3 Corollaries and Examples

The upper-class formulation produces a number of immediate special cases. We state representative consequences for the continuous versions; analogous statements follow from Theorems 2.3 and 2.6 by replacing continuity with $(H)$ and $(H')$.

For the Kannan-type results, write

$$M(x, y, z) = [S(x, x, Tx)]^p [S(y, y, Ty)]^q [S(z, z, Tz)]^{1-p-q}.$$

**Corollary 3.1.** *Let $(X, S)$ be a complete $S$-metric space and let $T : X \to X$. Suppose that there exist $\psi \in \Psi$, functions $\alpha_s, \mu_s : X^3 \to [0, \infty)$, integers $m, n \in \mathbb{N} \cup \{0\}$, a number $k > 0$, and positive $p, q$ with $p + q < 1$ such that*

$$[\alpha_s(x, y, z)]^n [S(Tx, Ty, Tz)]^k \leq [\mu_s(x, y, z)]^m [\psi(M(x, y, z))]^k,$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. Assume further that $T$ is continuous, $\alpha_s$-orbital admissible and $\mu_s$-orbital subadmissible, and that there exists $x_0 \in X$ with*

$$\alpha_s(x_0, x_0, Tx_0) \geq 1 \qquad \textit{and} \qquad \mu_s(x_0, x_0, Tx_0) \leq 1.$$

*Then $T$ has a fixed point in $X$.*

**Corollary 3.2.** *Let $(X, S)$ be a complete $S$-metric space and let $T : X \to X$. Suppose that there exist $\psi \in \Psi$, functions $\alpha_s, \mu_s : X^3 \to [0, \infty)$, and positive $p, q$ with $p + q < 1$ such that*

$$\alpha_s(x, y, z)S(Tx, Ty, Tz) \leq \mu_s(x, y, z)\psi(M(x, y, z)),$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. If $T$ is continuous, $\alpha_s$-orbital admissible and $\mu_s$-orbital subadmissible, and if there exists $x_0 \in X$ satisfying*

$$\alpha_s(x_0, x_0, Tx_0) \geq 1 \qquad \text{and} \qquad \mu_s(x_0, x_0, Tx_0) \leq 1,$$

*then $T$ has a fixed point in $X$.*

**Corollary 3.3.** *Under the hypotheses concerning $X, T, \psi, \alpha_s, \mu_s, p, q$ and $x_0$ in Corollary 3.2, assume instead that, for some $\ell > 1$,*

$$[S(Tx, Ty, Tz) + \ell]^{\alpha_s(x,y,z)} \leq \mu_s(x, y, z)\psi(M(x, y, z)) + \ell,$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. Then $T$ has a fixed point in $X$.*

**Corollary 3.4.** *Under the same standing hypotheses, assume that, for some $\ell > 0$,*

$$[\alpha_s(x, y, z) + \ell]^{S(Tx,Ty,Tz)} \leq (1 + \ell)^{\mu_s(x,y,z)\psi(M(x,y,z))},$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. Then $T$ has a fixed point in $X$.*

**Corollary 3.5.** *Under the same standing hypotheses, let $m, n \in \mathbb{N}$ and assume that*

$$\frac{m\alpha_s(x, y, z) + n}{m + n} S(Tx, Ty, Tz) \leq \mu_s(x, y, z)\psi(M(x, y, z)),$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. Then $T$ has a fixed point in $X$.*

**Remark 3.6.** In the preceding corollary, the formal choice $n = 0$ reduces the displayed condition to that of Corollary 3.2.

For the Čirić–Reich–Rus-type results, put

$$N(x, y, z) = [S(x, y, z)]^p[S(x, x, Tx)]^q[S(y, y, Ty)]^r[S(z, z, Tz)]^{1-p-q-r}.$$

**Corollary 3.7.** *Let $(X, S)$ be a complete $S$-metric space and let $T : X \to X$. Suppose that there exist $\psi \in \Psi$, functions $\alpha_s, \mu_s : X^3 \to [0, \infty)$, integers $m, n \in \mathbb{N} \cup \{0\}$, a number $k > 0$, and positive $p, q, r$ with $p + q + r < 1$ such that*

$$[\alpha_s(x, y, z)]^n[S(Tx, Ty, Tz)]^k \leq [\mu_s(x, y, z)]^m[\psi(N(x, y, z))]^k,$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. Assume further that $T$ is continuous, $\alpha_s$-orbital admissible and $\mu_s$-orbital subadmissible, and that there exists $x_0 \in X$ with*

$$\alpha_s(x_0, x_0, Tx_0) \geq 1 \qquad \text{and} \qquad \mu_s(x_0, x_0, Tx_0) \leq 1.$$

*Then $T$ has a fixed point in $X$.*

**Corollary 3.8.** *Let $(X, S)$ be a complete $S$-metric space and let $T : X \to X$. Suppose that there exist $\psi \in \Psi$, functions $\alpha_s, \mu_s : X^3 \to [0, \infty)$, and positive $p, q, r$ with $p + q + r < 1$ such that*

$$\alpha_s(x, y, z)S(Tx, Ty, Tz) \leq \mu_s(x, y, z)\psi(N(x, y, z)),$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. If $T$ is continuous, $\alpha_s$-orbital admissible and $\mu_s$-orbital subadmissible, and if there exists $x_0 \in X$ satisfying*

$$\alpha_s(x_0, x_0, Tx_0) \geq 1 \qquad \text{and} \qquad \mu_s(x_0, x_0, Tx_0) \leq 1,$$

*then $T$ has a fixed point in $X$.*

**Corollary 3.9.** *Under the standing hypotheses of Corollary 3.8, suppose that, for some $\ell > 1$,*

$$[S(Tx, Ty, Tz) + \ell]^{\alpha_s(x,y,z)} \leq \mu_s(x, y, z)\psi(N(x, y, z)) + \ell,$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. Then $T$ has a fixed point in $X$.*

**Corollary 3.10.** *Under the same standing hypotheses, suppose that, for some $\ell > 0$,*

$$[\alpha_s(x, y, z) + \ell]^{S(Tx,Ty,Tz)} \leq (1 + \ell)^{\mu_s(x,y,z)\psi(N(x,y,z))},$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. Then $T$ has a fixed point in $X$.*

**Corollary 3.11.** *Under the same standing hypotheses, let $m, n \in \mathbb{N}$ and assume that*

$$\frac{m\alpha_s(x, y, z) + n}{m + n} S(Tx, Ty, Tz) \leq \mu_s(x, y, z)\psi(N(x, y, z)),$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. Then $T$ has a fixed point in $X$.*

**Remark 3.12.** As above, the formal choice $n = 0$ reduces the preceding condition to that of Corollary 3.8.

Setting $\alpha_s \equiv 1$ and $\mu_s \equiv 1$ in Corollary 3.2 gives the next consequence.

**Corollary 3.13.** *Let $T$ be a continuous self-mapping on a complete $S$-metric space $(X, S)$. Suppose that, for some $\psi \in \Psi$ and positive $p, q$ with $p + q < 1$,*

$$S(Tx, Ty, Tz) \leq \psi\big([S(x, x, Tx)]^p[S(y, y, Ty)]^q[S(z, z, Tz)]^{1-p-q}\big),$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. Then $T$ has a fixed point in $X$.*

Taking $\alpha_s \equiv \mu_s \equiv 1$ and the special upper-class pair $h(x, y) = y$, $Q(s, t) = st$ in Theorem 2.5 yields the following.

**Corollary 3.14.** *Let $T$ be a continuous self-mapping on a complete $S$-metric space $(X, S)$. Suppose that, for some $\psi \in \Psi$ and positive $p, q, r$ with $p + q + r < 1$,*

$$S(Tx, Ty, Tz) \leq \psi\big([S(x, y, z)]^p[S(x, x, Tx)]^q[S(y, y, Ty)]^r \times [S(z, z, Tz)]^{1-p-q-r}\big),$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$. Then $T$ has a fixed point in $X$.*

**Corollary 3.15.** *Under the assumptions of Corollary 3.14, if $\psi(t) = \lambda t$ for some $\lambda \in [0, 1)$, then the condition becomes*

$$S(Tx, Ty, Tz) \leq \lambda [S(x, y, z)]^p [S(x, x, Tx)]^q [S(y, y, Ty)]^r \times [S(z, z, Tz)]^{1-p-q-r},$$

*and $T$ has a fixed point in $X$.*

**Remark 3.16.** The boundary choice $p = 1$ and $q = r = 0$ is not covered by the preceding corollary because its hypotheses require positive $p, q, r$ with $p + q + r < 1$. Formally, however, the displayed inequality reduces to the familiar Banach-type $S$-metric condition

$$S(Tx, Ty, Tz) \leq \lambda S(x, y, z).$$

Thus the present framework is naturally connected with the classical contraction principle, but the Banach case should not be read as a literal specialization of the preceding corollary.

**Corollary 3.17.** *Under the assumptions of Corollary 3.13, if $\psi(t) = \lambda t$ with $\lambda \in [0, 1)$, then*

$$S(Tx, Ty, Tz) \leq \lambda [S(x, x, Tx)]^p [S(y, y, Ty)]^q [S(z, z, Tz)]^{1-p-q},$$

*for all $x, y, z \in X \setminus \mathrm{Fix}(T)$, and $T$ has a fixed point in $X$.*

**Example 3.18.** Let $X = [0, 3]$ and define $S : X^3 \to [0, \infty)$ by

$$S(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Since $S(x, x, y) = 2|x - y|$, completeness of $(X, S)$ follows from completeness of the closed interval $[0, 3]$ in the usual metric. Define $T : X \to X$ by

$$T(x) = \begin{cases} \dfrac{1+x}{2}, & x \in [0, 1), \\ 1, & x \in [1, 3]. \end{cases}.$$

The two branches agree at $x = 1$, so $T$ is continuous. A direct computation gives

$$\mathrm{Fix}(T) = \{1\}.$$

Define

$$\alpha_s(x, y, z) = \begin{cases} 1, & x, y, z \in [1, 3], \\ 0, & \text{otherwise}, \end{cases} \qquad \mu_s(x, y, z) = \frac{1}{1 + S(x, y, z)}.$$

Take $x_0 = 3$. Since $T3 = 1$,

$$\alpha_s(3, 3, T3) = 1 \geq 1 \qquad \text{and} \qquad \mu_s(3, 3, T3) = \frac{1}{5} \leq 1.$$

If $x \in [1, 3]$, then $Tx = 1$ and $T^2 x = T1 = 1$, so

$$\alpha_s(Tx, Tx, T^2 x) = \alpha_s(1, 1, 1) = 1.$$

If $x \in [0,1)$, the implication defining $\alpha_s$-orbital admissibility is vacuous. Hence $T$ is $\alpha_s$-orbital admissible. Since $\mu_s \leq 1$ everywhere, $T$ is also $\mu_s$-orbital subadmissible.

Choose $\psi(t) = t/2$ and $p = q = 1/3$, and write

$$M(x,y,z) = [S(x,x,Tx)]^{1/3}[S(y,y,Ty)]^{1/3}[S(z,z,Tz)]^{1/3}.$$

For $x, y, z \in X \setminus \mathrm{Fix}(T)$, if at least one of $x, y, z$ lies outside $[1,3]$, then $\alpha_s(x,y,z) = 0$ and

$$\alpha_s(x,y,z)S(Tx,Ty,Tz) = 0 \leq \mu_s(x,y,z)\psi(M(x,y,z)).$$

If $x, y, z \in [1,3]$, then $Tx = Ty = Tz = 1$, so again

$$\alpha_s(x,y,z)S(Tx,Ty,Tz) = 0 \leq \mu_s(x,y,z)\psi(M(x,y,z)).$$

Thus all hypotheses of Corollary 3.2 are satisfied, and the fixed point guaranteed by the corollary is the unique point 1.

**Example 3.19.** Let $X = [0,5]$ with the same $S$-metric,

$$S(x,y,z) = |x-y| + |y-z| + |z-x|.$$

Define $T : X \to X$ by

$$T(x) = \begin{cases} x + \sin^2\left(\dfrac{\pi x}{2}\right), & x \in [0,1), \\ 2, & x \in [1,5]. \end{cases}.$$

The left-hand limit at 1 equals $2 = T1$, so $T$ is continuous. Its fixed point set is

$$\mathrm{Fix}(T) = \{0, 2\}.$$

Indeed, on $[0,1)$ the equation $Tx = x$ is equivalent to $\sin^2(\pi x/2) = 0$, which gives $x = 0$, while on $[1,5]$ it gives $x = 2$.

Define

$$\alpha_s(x,y,z) = \begin{cases} 1, & x,y,z \in [1,5], \\ 0, & \text{otherwise}, \end{cases} \qquad \mu_s(x,y,z) = 1.$$

Choose $x_0 = 5$. Then

$$\alpha_s(5,5,T5) = \alpha_s(5,5,2) = 1 \geq 1 \qquad \text{and} \qquad \mu_s(5,5,T5) = 1 \leq 1.$$

If $x \in [1,5]$, then $Tx = 2$ and $T^2x = 2$, so

$$\alpha_s(Tx,Tx,T^2x) = \alpha_s(2,2,2) = 1.$$

For $x \in [0,1)$, orbital admissibility is again vacuous whenever the antecedent vanishes. Thus $T$ is $\alpha_s$-orbital admissible, while $\mu_s \equiv 1$ makes $\mu_s$-orbital subadmissibility immediate.

Take $\psi(t) = t/2$ and, for instance, $p = q = r = 1/4$. Put

$$N(x,y,z) = [S(x,y,z)]^{1/4}[S(x,x,Tx)]^{1/4}[S(y,y,Ty)]^{1/4}[S(z,z,Tz)]^{1/4}.$$

If not all of $x, y, z$ belong to $[1, 5]$, then $\alpha_s(x, y, z) = 0$ and

$$\alpha_s(x, y, z)S(Tx, Ty, Tz) = 0 \le \mu_s(x, y, z)\psi(N(x, y, z)).$$

If $x, y, z \in [1, 5]$, then $Tx = Ty = Tz = 2$, and again

$$\alpha_s(x, y, z)S(Tx, Ty, Tz) = 0 \le \mu_s(x, y, z)\psi(N(x, y, z)).$$

Consequently, Corollary 3.8 applies. The mapping has the two distinct fixed points 0 and 2, showing that the fixed point conclusion need not be unique.

## 4 Conclusion

We introduced $(\alpha_s, \mu_s, (Q, h))$-interpolative Kannan- and Čirić–Reich–Rus-type contractions in complete $S$-metric spaces. The control functions $\alpha_s$ and $\mu_s$ permit the contractive condition to be localized along admissible orbits, while the upper-class pair $(Q, h)$ provides a flexible mechanism for generating concrete inequalities. Fixed point existence was obtained both under continuity and under the sequential conditions $(H)$ and $(H')$.

The examples illustrate two complementary features of the theory. Example 3.18 satisfies the hypotheses and has a unique fixed point, whereas Example 3.19 satisfies the corresponding Čirić–Reich–Rus-type assumptions and has two distinct fixed points. Thus the present results are principally existence theorems; additional hypotheses are required for uniqueness. It would be natural to investigate such uniqueness conditions, as well as coupled and common fixed point versions of the $(\alpha_s, \mu_s, (Q, h))$ framework.

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