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Research Article

Some Freeness Conditions for Semidualizing Modules over Cohen-Macaulay Local Rings

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Abstract

Abstract: Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let C be a semidualizing R-module such that $C^{\oplus n} \cong \mathrm{Hom}_R(M, R)$ for some finitely generated R-module M and some positive integer n. We give conditions on M and C that imply that C is free. In particular, if R is a Cohen-Macaulay local ring of dimension at most one, then $C \cong R$.
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1 Introduction

Throughout the paper, $(R, \mathfrak{m})$ denotes a commutative Noetherian local ring, and all modules are finitely generated unless otherwise stated. An R-module C is called *semidualizing* if

$$\mathrm{Hom}_R(C, C) \cong R \qquad \text{and} \qquad \mathrm{Ext}^i_R(C, C) = 0 \quad \text{for all } i > 0.$$

The module R itself is always semidualizing. This observation has motivated the extension of many classical homological notions from R to an arbitrary semidualizing module. For example, Takahashi and White [10] study C-projective and C-injective dimensions, while Holm and Jørgensen [7] investigate C-Gorenstein homological dimensions.

Many characterizations of rings and modules can be formulated in terms of semidualizing modules, illustrating their importance. For instance, it is shown in [10] that if the residue field $R/\mathfrak{m}$ has finite C-projective dimension, then R is regular. A central question in this area is to determine

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conditions on a semidualizing module C that force $C \cong R$; see, for example, [2,5,7]. Bagheri and Taherizadeh [2] proved that if R is a one-dimensional Cohen-Macaulay local ring and C is semidualizing with $C \cong M^* = \mathrm{Hom}_R(M,R)$ for some finitely generated R-module M, then C is free. Another result, [5, Theorem 5.6], states that if R is Cohen-Macaulay and a semidualizing module C has rank, then C is free if and only if $\mathrm{Ext}^i_R(C,R) = 0$ for all $i > 0$.

In this paper, we consider the more general situation

$$M^* \cong C^{\oplus n},$$

for some positive integer n and seek conditions that imply $C \cong R$. In particular, Theorem 2.8 extends [2, Theorem 4.9]. We also derive consequences concerning Gorenstein local rings.

2 Preliminaries

The number of minimal generators of an R-module M is denoted by $\mu(M)$. The type of M is denoted by $r_R(M)$ and is defined by

$$r_R(M) = \dim_{R/\mathfrak{m}} \mathrm{Ext}_R^{\mathrm{depth}_R M}(R/\mathfrak{m}, M).$$

If $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module of rank r for every $\mathfrak{p} \in \mathrm{Ass}\, R$, we say that M has rank r and write $\mathrm{rank}(M) = r$.

Definition 2.1. An R-module C is called a *semidualizing R-module* if $\mathrm{Hom}_R(C,C) \cong R$ and $\mathrm{Ext}^i_R(C,C) = 0$ for all $i > 0$.

Definition 2.2. A semidualizing R-module D is called *dualizing* if $\mathrm{id}_R D < \infty$. When R is Cohen-Macaulay, such a module is a canonical module and is commonly denoted by ω_R.

Remark 2.3. If C is semidualizing and ω_R is a canonical module of R, then $\mathrm{Hom}_R(C, \omega_R)$ is semidualizing by [11, Corollary 4.1.3]. We write

$$C^\vee = \mathrm{Hom}_R(C, \omega_R).$$

We recall the Foxby classes associated with a semidualizing module.

Definition 2.4. Let C be a semidualizing R-module. The *Auslander class* $\mathcal{A}_C(R)$ consists of all R-modules M satisfying:

(i) the natural map

$$\gamma_M : M \longrightarrow \mathrm{Hom}_R(C, C \otimes_R M), \qquad \gamma_M(m)(c) = c \otimes m,$$

is an isomorphism;

(ii) $\mathrm{Tor}_i^R(C,M) = 0 = \mathrm{Ext}^i_R(C, C \otimes_R M)$ for every $i \geq 1$.

The *Bass class* $\mathcal{B}_C(R)$ consists of all R-modules M satisfying:

(i) the evaluation map

$$\zeta_M : C \otimes_R \mathrm{Hom}_R(C, M) \longrightarrow M, \qquad \zeta_M(c \otimes f) = f(c),$$

is an isomorphism;

(ii) $\mathrm{Ext}^i_R(C, M) = 0 = \mathrm{Tor}^R_i(C, \mathrm{Hom}_R(C, M))$ for every $i \geq 1$.

Lemma 2.5. *If R is local, then every semidualizing R-module is indecomposable.*

Proof. See [9, Proposition 2.2]. ∎

Lemma 2.6. *Let R be local and let C be a semidualizing R-module. Suppose that*

$$C^{\oplus n} \cong M \oplus N,$$

for some positive integer n and some R-modules M and N. Then there exist nonnegative integers s and t such that

$$M \cong C^{\oplus s} \qquad \text{and} \qquad N \cong C^{\oplus t}.$$

Proof. Applying $\mathrm{Hom}_R(C, -)$ gives

$$\mathrm{Hom}_R(C, M) \oplus \mathrm{Hom}_R(C, N) \cong \mathrm{Hom}_R(C, C^{\oplus n}) \cong R^{\oplus n}.$$

Because R is local, both $\mathrm{Hom}_R(C, M)$ and $\mathrm{Hom}_R(C, N)$ are free. The conclusion follows from [2, Lemma 3.5]. ∎

Proposition 2.7. *Let B and C be semidualizing R-modules. Then the following conditions are equivalent:*

(i) $B \in \mathcal{A}_C(R)$;

(ii) $C \in \mathcal{A}_B(R)$.

Moreover, if these conditions hold, then $C \otimes_R B$ is a semidualizing R-module.

Proof. It is enough to prove (i)⇒(ii). Assume that $B \in \mathcal{A}_C(R)$. Then

$$B \cong \mathrm{Hom}_R(C, C \otimes_R B).$$

Using Hom–tensor adjointness, we obtain

$$\begin{aligned} R &\cong \mathrm{Hom}_R(B, B) \\ &\cong \mathrm{Hom}_R\big(B, \mathrm{Hom}_R(C, C \otimes_R B)\big) \\ &\cong \mathrm{Hom}_R\big(C, \mathrm{Hom}_R(B, B \otimes_R C)\big). \end{aligned}$$

Therefore,

$$C \cong \mathrm{Hom}_R(B, B \otimes_R C).$$

It remains to verify

$$\mathrm{Tor}^R_i(B, C) = 0 \qquad \text{and} \qquad \mathrm{Ext}^i_R(B, B \otimes_R C) = 0 \quad (i > 0).$$

Using the hypothesis and [11, Lemma 3.1.13(c)], we obtain

$$\operatorname{Tor}_i^R(B,C) \cong \operatorname{Tor}_i^R(C\otimes_R B, \operatorname{Hom}_R(C,C)) \cong \operatorname{Tor}_i^R(C\otimes_R B, R) = 0,$$

and

$$\operatorname{Ext}_R^i(B, C\otimes_R B) \cong \operatorname{Ext}_R^i(R\otimes_R B, C\otimes_R B) \cong \operatorname{Ext}_R^i(R,C) = 0.$$

Thus $C \in \mathcal{A}_B(R)$. The final assertion follows from the same Foxby-class argument. ■

Theorem 2.8. *Let R be a Cohen-Macaulay local ring with $\dim R \leq 1$. Suppose that C is a semidualizing R-module and M is a finitely generated R-module such that*

$$M^* \cong C^{\oplus n},$$

for some positive integer n. Then $C \cong R$.

Proof. We may assume that R is complete; hence a canonical module $\omega = \omega_R$ exists. If $M \cong R^{\oplus t}$ for some positive integer t, then

$$R^{\oplus t} \cong M^* \cong C^{\oplus n}.$$

By Lemmas 2.6 and 2.5, this implies $C \cong R$.

Assume now that M is not free. Choose a finite presentation

$$R^{\oplus s} \longrightarrow R^{\oplus t} \longrightarrow M \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(-,R)$ gives an exact sequence

$$0 \longrightarrow M^* \longrightarrow R^{\oplus t} \xrightarrow{f} R^{\oplus s}.$$

Set $L = \operatorname{im} f$. Then

$$0 \longrightarrow M^* \longrightarrow R^{\oplus t} \longrightarrow L \longrightarrow 0. \qquad (*)$$

If $L = 0$, then $C^{\oplus n} \cong M^* \cong R^{\oplus t}$, and hence $C \cong R$.

Suppose $L \neq 0$. Since L is a submodule of a free module and $\operatorname{depth} R \leq 1$, the depth lemma gives $\operatorname{depth}_R L = \operatorname{depth} R$, so L is maximal Cohen-Macaulay. Since $\omega \in \mathcal{B}_C(R)$, [11, Theorem 3.2.1(b)] gives $C^\vee \in \mathcal{A}_C(R)$. Because $C^\vee$ is semidualizing, Proposition 2.7 yields $C \in \mathcal{A}_{C^\vee}(R)$. Also $R \in \mathcal{A}_{C^\vee}(R)$; using $(*)$, we obtain $L \in \mathcal{A}_{C^\vee}(R)$. Hence, for every $i \geq 1$,

$$\operatorname{Tor}_i^R(C^\vee, L) = 0, \qquad \operatorname{Ext}_R^i(C^\vee, C^\vee \otimes_R L) = 0, \qquad \operatorname{Hom}_R(C^\vee, C^\vee\otimes_R L) \cong L.$$

Tensoring $(*)$ with $C^\vee$ gives

$$0 \longrightarrow C^\vee \otimes_R M^* \longrightarrow (C^\vee)^{\oplus t} \longrightarrow C^\vee \otimes_R L \longrightarrow 0. \qquad (**)$$

By [1, Lemma 4.1],

$$\operatorname{depth}_R(C^\vee \otimes_R L) = \operatorname{depth}_R \operatorname{Hom}_R(C^\vee, C^\vee \otimes_R L) = \operatorname{depth}_R L = \operatorname{depth} R.$$

Thus $C^\vee \otimes_R L$ is maximal Cohen-Macaulay. Applying $\operatorname{Hom}_R(-,\omega)$ to $(**)$ yields

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_R(C^\vee \otimes_R L, \omega) \longrightarrow \operatorname{Hom}_R((C^\vee)^{\oplus t}, \omega) \\ \longrightarrow \operatorname{Hom}_R(C^\vee \otimes_R M^*, \omega) \longrightarrow \operatorname{Ext}_R^1(C^\vee \otimes_R L, \omega). \end{aligned}$$

The last term is zero because $C^\vee \otimes_R L$ is maximal Cohen-Macaulay. Moreover,

$$\begin{aligned}\operatorname{Hom}_R(C^\vee \otimes_R M^*, \omega) &\cong \operatorname{Hom}_R(C^\vee \otimes_R C^{\oplus n}, \omega)\\ &\cong \operatorname{Hom}_R(C^\vee \otimes_R C, \omega)^{\oplus n}\\ &\cong \operatorname{Hom}_R(\omega, \omega)^{\oplus n} \cong R^{\oplus n},\end{aligned}$$

and

$$\operatorname{Hom}_R((C^\vee)^{\oplus t}, \omega) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(C, \omega), \omega)^{\oplus t} \cong C^{\oplus t}.$$

Consequently, there is a split exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(C^\vee \otimes_R L, \omega) \longrightarrow C^{\oplus t} \longrightarrow R^{\oplus n} \longrightarrow 0,$$

so

$$C^{\oplus t} \cong R^{\oplus n} \oplus \operatorname{Hom}_R(C^\vee \otimes_R L, \omega).$$

By [11, Proposition 3.1.6(b)] and [11, Corollary 4.1.1], we conclude that $C \cong R$. ■

Corollary 2.9. *Let R be a local ring with $\dim R \leq 1$. Suppose that C is a semidualizing R-module and M is a finitely generated R-module with $\operatorname{id}_R M < \infty$. If*

$$M^* \cong C^{\oplus n},$$

for some positive integer n, then R is Gorenstein.

Proof. By [3, Remark 9.6.4], R is Cohen-Macaulay. Hence Theorem 2.8 gives $C \cong R$. We may assume that R is complete, and write $\omega = \omega_R$. Since $\operatorname{id}_R M < \infty$, one has $M \in \mathcal{B}_\omega(R)$ and therefore

$$M \cong \omega \otimes_R \operatorname{Hom}_R(\omega, M).$$

Set $L = \operatorname{Hom}_R(\omega, M)$. Then

$$R^{\oplus n} \cong M^* \cong \operatorname{Hom}_R(\omega \otimes_R L, R) \cong \operatorname{Hom}_R(\omega, L^*).$$

By [2, Lemma 3.5], $L^* \cong \omega^{\oplus n}$. By [4, Lemma 1.1.9(a)], there exists a finitely generated R-module N such that

$$(\omega^{**})^{\oplus n} \cong \omega^{\oplus n} \oplus N.$$

Thus

$$\operatorname{Ass} N \subseteq \operatorname{Ass} \omega^{**} \subseteq \operatorname{Ass} R.$$

Let $\mathfrak{p} \in \operatorname{Ass} R$. Localizing gives

$$(\omega_\mathfrak{p}^{**})^{\oplus n} \cong \omega_\mathfrak{p}^{\oplus n} \oplus N_\mathfrak{p}.$$

Since $M_\mathfrak{p}^* \cong R_\mathfrak{p}^{\oplus n}$ and $\operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p} = 0$, the module $M_\mathfrak{p}$ is injective and maximal Cohen-Macaulay; hence $M_\mathfrak{p} \cong \omega_\mathfrak{p}^{\oplus t}$ for some $t > 0$; see [3, Exercise 3.2.28]. Therefore $(\omega_\mathfrak{p}^*)^{\oplus n} \cong R_\mathfrak{p}^{\oplus n}$, so $\omega_\mathfrak{p}^*$ is free. Write $\omega_\mathfrak{p}^* \cong R_\mathfrak{p}^{\oplus s}$. Comparing ranks in

$$R_\mathfrak{p}^{\oplus sn} \cong R_\mathfrak{p}^{\oplus n},$$

gives $s = 1$. Thus $\omega_{\mathfrak{p}}^{*} \cong R_{\mathfrak{p}}$, and [2, Lemma 3.5] yields $\omega_{\mathfrak{p}} \cong R_{\mathfrak{p}}$. It follows that $N_{\mathfrak{p}} = 0$ for every $\mathfrak{p} \in \operatorname{Ass} R$. Hence $\operatorname{Ass} N = \varnothing$, so $N = 0$ and

$$(\omega^{**})^{\oplus n} \cong \omega^{\oplus n}.$$

By Lemma 2.6, $\omega^{**} \cong \omega^{\oplus r}$ for some positive integer r. Therefore

$$\omega^{\oplus nr} \cong \omega^{\oplus n},$$

so $r = 1$ and $\omega^{**} \cong \omega$. By [6, Corollary 7.29], R is Gorenstein. ∎

In the following result, we extend [2, Theorem 4.1].

Theorem 2.10. *Let R be a Cohen-Macaulay local ring with $\dim R = d$, let C be a semidualizing R-module, and let M be a finitely generated R-module such that*

$$M^{*} \cong C^{\oplus n},$$

for some positive integer n. Then the following statements hold.

(i) If $\operatorname{Ext}_R^i(M, R) = 0$ for every $1 \le i \le d$, then $C \cong R$ and $M \cong R^{\oplus n}$.

(ii) If $\operatorname{Ext}_R^i(C, R) = 0$ for every $1 \le i \le d$, then $C \cong R$.

Proof. If $\mathfrak{p} \in \operatorname{Ass} R$, then $\operatorname{depth} R_{\mathfrak{p}} = \dim R_{\mathfrak{p}} = 0$ and

$$(M_{\mathfrak{p}})^{*} \cong C_{\mathfrak{p}}^{\oplus n}.$$

By Theorem 2.8, applied to $R_{\mathfrak{p}}$, we have $C_{\mathfrak{p}} \cong R_{\mathfrak{p}}$. Hence C has rank one.

Proof of (i). By [3, Exercise 1.4.23],

$$\operatorname{rank}(M) = \operatorname{rank}(M^{*}) = \operatorname{rank}(C^{\oplus n}) = n \operatorname{rank}(C) = n.$$

By [5, Lemma 2.1],

$$n r_R(C) = r_R(M^{*}) = \mu(M) r(R) = \mu(M)\mu(C) r_R(C),$$

and therefore $\mu(M)\mu(C) = n$. Put $s = \mu(M)$. If $s < n$, a minimal presentation gives an exact sequence

$$0 \longrightarrow L \longrightarrow R^{\oplus s} \longrightarrow M \longrightarrow 0. \tag{$\dagger$}$$

Since M and R have rank, [3, Proposition 1.4.5] implies that L has rank and

$$\operatorname{rank}(L) = s - n < 0,$$

a contradiction. Thus $s = n$, so $\mu(M) = n$ and $\mu(C) = 1$. Since a semidualizing module is faithful, $C \cong R$. Now $s = n$ gives $\operatorname{rank}(L) = 0$. Localizing (†) at $\mathfrak{p} \in \operatorname{Ass} R$ yields $L_{\mathfrak{p}} = 0$. Hence $\operatorname{Ass} L \cap \operatorname{Ass} R = \varnothing$. On the other hand, (†) gives $\operatorname{Ass} L \subseteq \operatorname{Ass} R$, so $L = 0$. Therefore $M \cong R^{\oplus n}$.

Proof of (ii). Since C has rank, [5, Corollary 4.5(2)] gives $C \cong R$. ∎

Theorem 2.11. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with* $\operatorname{depth} R = d$. *The following statements are equivalent.*

(i) There exist a semidualizing R-module C, a finitely generated R-module M of finite injective dimension, and a positive integer t such that

$$M^* \cong C^{\oplus t},$$

and

$$\operatorname{Ext}_R^i(M, R) = 0 \qquad (1 \leq i \leq d).$$

(ii) $\operatorname{Ext}_R^{d+1}(k, R) = 0$.

(iii) R is Gorenstein.

Proof. (i)⇒(iii). Since M is a nonzero finitely generated module of finite injective dimension, R is Cohen–Macaulay; see [3, Remark 9.6.4]. Thus $\dim R = d$. Theorem 2.10(i) gives

$$C \cong R \qquad \text{and} \qquad M \cong R^{\oplus t}.$$

Since $\operatorname{id}_R M < \infty$, it follows that $\operatorname{id}_R R < \infty$, so R is Gorenstein.

(ii)⇒(iii). This is the classical Bass vanishing criterion; see [8, Section 18].

(iii)⇒(i). Take $C = R$, $M = R$, and $t = 1$. Then $M^* \cong R$, $\operatorname{id}_R R < \infty$, and $\operatorname{Ext}_R^i(R, R) = 0$ for every $i > 0$. ∎

Corollary 2.12. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R. Assume that*

$$\operatorname{Ext}_R^i(\omega_R, R) = 0 \qquad (1 \leq i \leq d).$$

If R is generically Gorenstein, then R is Gorenstein.

Proof. Since R is generically Gorenstein, $(\omega_R)_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Ass} R$. Thus ω_R has rank one. Applying Theorem 2.10(ii) with $C = \omega_R$, we obtain $\omega_R \cong R$. Hence R is Gorenstein. ∎

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