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Research Article

Some Algebraic Aspects of Generalized Hukuhara Differentiability of Fuzzy-valued Functions

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Abstract

Abstract: This paper investigates some algebraic aspects of the generalized Hukuhara differentiability (gHD) of fuzzy-valued functions (FVFs) based on the generalized Hukuhara difference (gH-difference) using α-cuts. It starts by describing some operations on real compact intervals and fuzzy numbers (FNs) through their α-cuts when the α-cut method converts FNs to real compact intervals. And establishing the necessary conditions for the existence of the gH-difference between two FNs. The paper examines two distinct FNs: trapezoidal fuzzy numbers (TrFNs) and triangular fuzzy numbers (TFNs). The concept of the gH-difference is then extended to FVFs, thereby facilitating the definition of gHD for such functions. The paper investigates whether the gH-difference, summation, subtraction and scalar multiplication of gHD FVFs are themselves gHD. Furthermore, it provides practical methods for deriving the gHD of these FVFs.
Keywords: Generalized Hukuhara differentiability, Fuzzy-valued functions, α-cut set, Operations on real compact intervals, Fuzzy numbers.
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1 Introduction

In 1965, Zadeh [1] presented the concept of fuzzy sets (FSs), which serve as an effective tool for describing uncertainty and managing ambiguous or subjective information within mathematical contexts. This concept has developed in different ways and is used to solve many real-world issues, such as the golden mean [2], particle systems [3], quantum optics and gravity [4], chaotic systems [5], medicine [6, 7] and engineering problems [8].

Interval analysis is a mathematical field introduced in the writings of Moore [9] and Sunaga [10]. It has garnered significant attention for offering mathematical tools that facilitate the modelling

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and management of situations involving interval type uncertainties. However, some problems with the "difference" operation make it difficult to create a clear differential theory in interval spaces, leading to the development of different types of algebraic arithmetic in this area; [11, 12, 13].

Stefanini [14] proposed the gH-difference. Applying the gH-difference, the authors in [15] introduced a gHD for IVFs of a real variable. The gHD is a valuable notion in fuzzy theory; this concept of gHD has been applied across various fields, see [16, 17, 18, 19, 20, 21]; and gHD has proved its efficacy as a powerful instrument with many applications in interval and fuzzy function spaces.

This paper aims to show when the gH-difference, summation, subtraction and scalar multiplication of two gHD FVFs is gHD.

We structure the paper in the following manner: Section 2 delineates some arithmetic operations on real compact intervals. Section 3 presents some notations about FSs and FNs. In Section 4, we introduce the concept of gHD for FVFs based on gH-difference. The discourse on conclusions is presented in Section 5.

2 Preliminaries

This section presents the fundamental concepts, useful results and essential notation used throughout the paper. We first define what a compact real interval is.

Definition 2.1. *Let $\underline{m}, \overline{m} \in \mathbb{R}$ such that $\underline{m} \leq \overline{m}$, an interval $[\underline{m}, \overline{m}]$ is a closed and bounded (compact) nonempty real interval, that is*

$$\mathcal{M} = [\underline{m}, \overline{m}] = \{x \in \mathbb{R} \mid \underline{m} \leq x \leq \overline{m}\}.$$

We indicate by $I_{\mathbb{R}}$ the collection of all compact intervals in $\mathbb{R}$, described as follows:

$$I_{\mathbb{R}} = \{[\underline{m}, \overline{m}] \mid \underline{m}, \overline{m} \in \mathbb{R}\}.$$

Given two intervals $\mathcal{M} = [\underline{m}, \overline{m}]$, $\mathcal{N} = [\underline{n}, \overline{n}]$ and $\lambda \in \mathbb{R}$, we have these operations (for $\mathcal{M}/\mathcal{N} \in I_{\mathbb{R}},\ 0 \notin \mathbb{N}$) [22]:

$$\mathcal{M} + \mathcal{N} = [\underline{m} + \underline{n},\ \overline{m} + \overline{n}], \tag{1}$$
$$\mathcal{M} - \mathcal{N} = [\underline{m} - \overline{n},\ \overline{m} - \underline{n}], \tag{2}$$
$$\mathcal{M}\mathcal{N} = [\min(\mathcal{P}),\ \max(\mathcal{P})], \tag{3}$$
$$\mathcal{M}/\mathcal{N} = [\min(\mathcal{Q}),\ \max(\mathcal{Q})], \tag{4}$$
$$\lambda\mathcal{M} = \begin{cases} [\lambda\underline{m},\ \lambda\overline{m}], \text{ if } \lambda \geq 0, \\ [\lambda\overline{m},\ \lambda\underline{m}], \text{ if } \lambda < 0, \end{cases} \tag{5}$$

where $\mathcal{P} = \{\underline{m}\ \underline{n}, \underline{m}\ \overline{n}, \overline{m}\ \underline{n}, \overline{m}\ \overline{n}\}$ and $\mathcal{Q} = \{\underline{m}/\underline{n}, \underline{m}/\overline{n}, \overline{m}/\underline{n}, \overline{m}/\overline{n}\}$.

Another operation on real compact intervals that solves the problem $\mathcal{M} - \mathcal{M} \neq 0$ is gH-difference, which is defined as follows [23]:

Definition 2.2. *Let $\mathcal{M}, \mathcal{N} \in I_{\mathbb{R}}$, the gH-difference of $\mathcal{M}$ and $\mathcal{N}$ is the interval $\mathcal{W} \in I_{\mathbb{R}}$ such that*

$$\mathcal{W} = \mathcal{M} \ominus_{\mathrm{gH}} \mathcal{N} \iff \begin{cases} \quad (i)\ \mathcal{M} = \mathcal{N} + \mathcal{W}, \\ or \quad (ii)\ \mathcal{N} = \mathcal{M} + (-1)\mathcal{W}. \end{cases} \tag{6}$$

The gH-difference operation between two real compact intervals always exists and can be written as follows:

$$\mathcal{W} = \mathcal{M} \ominus_{\text{gH}} \mathcal{N} = [\min\{\underline{m} - \underline{n}, \overline{m} - \overline{n}\}, \max\{\underline{m} - \underline{n},\ \overline{m} - \overline{n}\}]. \tag{7}$$

The following properties were obtained by [24, 25].

Proposition 2.3. *Given* $\mathcal{M}, \mathcal{N} \in I_{\mathbb{R}}$ *and* $\lambda \in \mathbb{R}$.

1) $\mathcal{M} \ominus_{\text{gH}} \mathcal{M} = 0 \neq \mathcal{M} - \mathcal{M}$.

2). $\lambda(\mathcal{M} \ominus_{\text{gH}} \mathcal{N}) = \lambda\mathcal{M} \ominus_{\text{gH}} \lambda\mathcal{N}$.

3). $\mathcal{M} \ominus_{\text{gH}} \mathcal{N} = (-\mathcal{N}) \ominus_{\text{gH}} (-\mathcal{M}) = -(\mathcal{N} \ominus_{\text{gH}} \mathcal{M})$.

4). $(\mathcal{M} + \mathcal{N}) \ominus_{\text{gH}} \mathcal{N} = \mathcal{M}$ *and* $\mathcal{M} \ominus_{\text{gH}} (\mathcal{M} - \mathcal{N}) = \mathcal{N}$.

3 Notation on Fuzzy numbers

In classical set theory, the inclusion of elements in a set is ascertained by a binary method. Conversely, FS theory facilitates a more nuanced evaluation of membership through a membership function (MF) that yields values within the interval $[0, 1]$. The concept is exemplified by the subsequent array of definitions.

Definition 3.1. *[26] A FS* $\widetilde{\mathcal{M}}$ *on a space (universe)* $X \in \mathbb{R}^n$ *of elements is a set of ordered pairs,*

$$\widetilde{\mathcal{M}} = \{(t, \widetilde{\mathcal{M}}(x)) \mid t \in X,\ \widetilde{\mathcal{M}}(t) \in [0, 1]\},$$

where

$$\widetilde{\mathcal{M}} : \mathbb{R}^n \to [0, 1].$$

and $\widetilde{\mathcal{M}}(x)$ *is the MF that maps each element* t *to its membership value (MV).*

Example 3.2. *Suppose* $X = \{1, 2, 3, 4, 5, 6, 7\}$. *Consider the FS* $\widetilde{\mathcal{M}}$ *as the set of numbers closer to* 5. *Now take the MVs as* $\widetilde{\mathcal{M}}(1) = 0,\ \widetilde{\mathcal{M}}(2) = 0.2,\ \widetilde{\mathcal{M}}(3) = 0.5,\ \widetilde{\mathcal{M}}(4) = 0.8,\ \widetilde{\mathcal{M}}(5) = 1,\ \widetilde{\mathcal{M}}(6) = 0.8$ *and* $\widetilde{\mathcal{M}}(7) = 0.5$. *Then we write the FS* $\widetilde{\mathcal{M}}$ *as*

$$\widetilde{\mathcal{M}} = \{(1, 0), (2, 0.2), (3, 0.5), (4, 0.8), (5, 1), (6, 0.8), (7, 0.5)\}.$$

Remark 3.3. *[23] The support of* $\widetilde{\mathcal{M}}$ *is* $Support(\widetilde{\mathcal{M}}) = \{t \mid t \in X, \widetilde{\mathcal{M}}(t) > 0\}$. *We define* $[\widetilde{\mathcal{M}}]_{(\alpha)} = \{t \mid t \in X, \widetilde{\mathcal{M}}(t) \geqslant \alpha\}$ *the* α*-level set (or simply the* α*-cut) of* $\widetilde{\mathcal{M}}$, *with* $0 < \alpha \leqslant 1$ *and* $[\widetilde{\mathcal{M}}]_0 = cl\{supp(\widetilde{\mathcal{M}})\}$ *where* $cl(A)$ *denotes the closure of the subset* $A \subset \mathbb{R}^n$. *The core of* $\widetilde{\mathcal{M}}$ *is given by* $Core(\widetilde{\mathcal{M}}) = \{t \mid t \in X,\ \widetilde{\mathcal{M}}(t) = 1\}$.

A FN is a specific type of FS, characterised by a unimodal and normal MF. Zadeh [1] presented FNs as a pragmatic method for effectively managing indefinite numerical values.

Definition 3.4. *[26] A FS* $\widetilde{\mathcal{M}}$ *on* $\mathbb{R}$ *is a FN if it satisfies the following conditions:*

1). $\widetilde{\mathcal{M}}$ *is normal FS. i.e,* $\exists\lambda \in \mathbb{R}$ *for which* $\widetilde{\mathcal{M}}(\lambda) = 1$.

2). $[\widetilde{\mathcal{M}}]_0$ *is compact.*

3). The MF $\widetilde{\mathcal{M}}(x)$ *is piecewise continuous.*

4). $\widetilde{\mathcal{M}}$ *is convex. i.e,* $\widetilde{\mathcal{M}}(\beta\lambda + (1-\beta)\gamma) \geqslant \min(\widetilde{\mathcal{M}}(\lambda), \widetilde{\mathcal{M}}(\gamma))$ *for any* $\lambda, \gamma \in \mathbb{R}$ *and* $\beta \in [0, 1]$.

Definition 3.5. *[8] A TrFN denoted by* $\widetilde{\mathcal{M}} = (\lambda, \gamma, \mu, \rho)$*, where* $\lambda < \gamma < \mu < \rho$ *and are real numbers and its MF* $\widetilde{\mathcal{M}}(x)$ *is denoted as (shown in the Figure 1)*

$$\widetilde{\mathcal{M}}(x) = \begin{cases} (x-\lambda)/(\gamma-\lambda), & \text{when } \lambda \leq x \leq \gamma, \\ 1, & \text{when } \gamma \leq x \leq \mu, \\ (\rho-x)/(\rho-\mu), & \text{when } \mu \leq x \leq \rho, \\ 0, & \text{otherwise }, \end{cases}$$

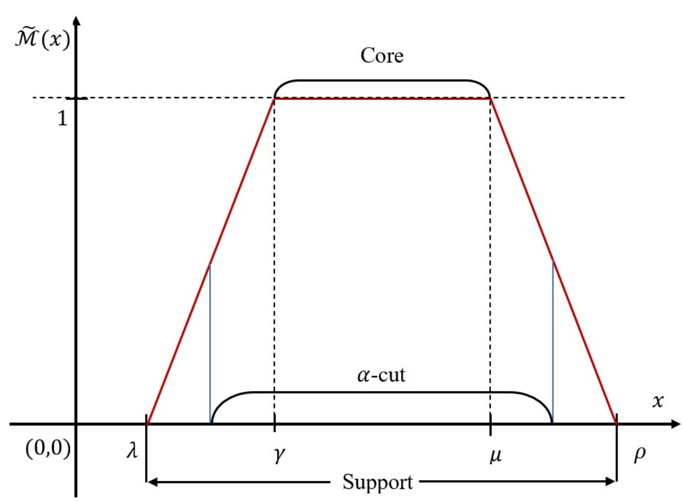


Figure 1: TrFN with its Support, α-cut and Core.

$\widetilde{\mathcal{M}}$ *is a TFN if* $\gamma = \mu$*, (shown in the Figure 2).*

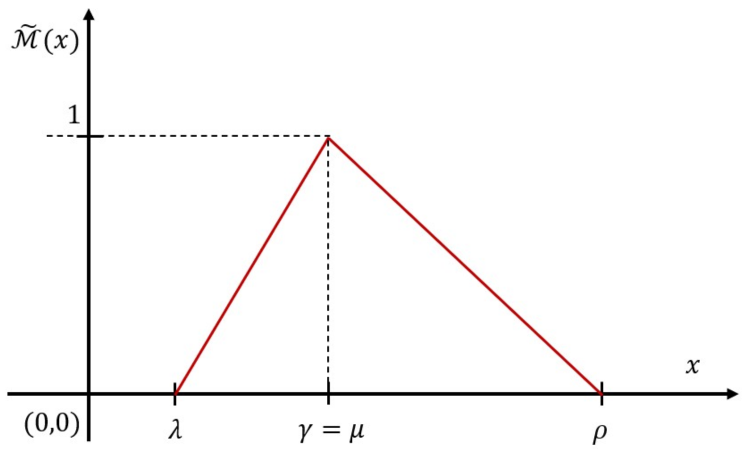


Figure 2: TFN.

Remark 3.6. *We have several types of TrFNs (TFNs when $\gamma = \mu$), listed below:*

1). $\widetilde{\mathcal{M}} = (\lambda, \gamma, \mu, \rho)$ *is a positive TrFN (PTrFN) if* $\lambda > 0$.

2). $\widetilde{\mathcal{M}} = (\lambda, \gamma, \mu, \rho)$ *is a negative TrFN (NTrFN) if* $\rho < 0$.

3). A TrFN $\widetilde{\mathcal{M}} = (\lambda, \gamma, \mu, \rho)$ *is partial negative (PNTrFN), where at least one or two or three components of* $\widetilde{\mathcal{M}}$ *are negative and not all.*

Let $\mathcal{F}_{\mathcal{R}}$ be the space of all FNs and for every $\alpha \in [0, 1]$ the α-cut of any TrFN $\widetilde{\mathcal{M}} = (\lambda, \gamma, \mu, \rho)$ constitute a nonempty compact interval of the form

$$\widetilde{\mathcal{M}}_\alpha = [\lambda + \alpha(\gamma - \lambda), \rho - \alpha(\rho - \mu)] \tag{8}$$

[24]

Remark 3.7. *For all $\alpha \in [0, 1]$ the necessary conditions for $\widetilde{\mathcal{W}} = \widetilde{\mathcal{M}} \ominus_{\text{gH}} \widetilde{\mathcal{N}} \in \mathcal{F}_{\mathcal{R}}$ are as follows:*

$$[\widetilde{\mathcal{W}}]_\alpha = [\underline{w}_\alpha, \overline{w}_\alpha] = \begin{cases} & (a)\ \left[\underline{m}_\alpha - \underline{n}_\alpha, \overline{m}_\alpha - \overline{n}_\alpha\right], \\ or & (b)\ \left[\overline{m}_\alpha - \overline{n}_\alpha, \underline{m}_\alpha - \underline{n}_\alpha\right], \end{cases} \tag{9}$$

where in both (a) and (b), $\underline{w}_\alpha$ is increasing, $\overline{w}_\alpha$ is decreasing and $\underline{w}_\alpha \leq \overline{w}_\alpha$.

Example 3.8. *If we have two PTrFNs $\widetilde{\mathcal{M}} = (3, 4, 6, 7)$ and $\widetilde{\mathcal{N}} = (2, 4, 6, 9)$.For any $\alpha \in [0, 1]$, the α-cut of $\widetilde{\mathcal{M}}$ is $[\widetilde{\mathcal{M}}]_\alpha = [3+\alpha, 7-\alpha]$ and α-cut of $\widetilde{\mathcal{N}}$ is $[\widetilde{\mathcal{N}}]_\alpha = [2+2\alpha, 9-3\alpha]$; then $\underline{m}_\alpha - \underline{n}_\alpha = 1-\alpha$ and $\overline{m}_\alpha - \overline{n}_\alpha = -2 + 2\alpha$. Since $\underline{m}_\alpha - \underline{n}_\alpha \geq \overline{m}_\alpha - \overline{n}_\alpha$ and $\underline{m}_\alpha - \underline{n}_\alpha$ is decreasing and $\overline{m}_\alpha - \overline{n}_\alpha$ is increasing as required by Equation (9)-(b). Consequently, the gH-difference exists, and it is $\widetilde{\mathcal{W}} = [-2 + 2\alpha, 1 - \alpha]$, as illustrated in Figure 3.*

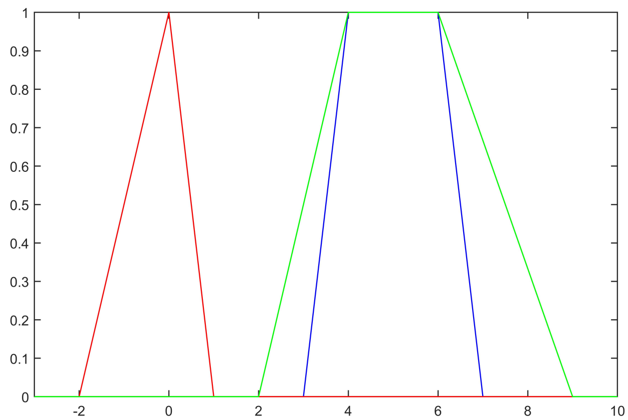


Figure 3: The gH-difference (red lines) of TrFNs $\widetilde{\mathcal{M}}$ (blue lines) and $\widetilde{\mathcal{N}}$ (green lines).

4 gHD of FVFs

A function $\widetilde{\mathcal{S}} : \mathbb{R} \to \mathcal{F}_{\mathcal{R}}$ is referred to as a FVF. For each $\alpha \in [0,1]$, corresponding to $\widetilde{\mathcal{S}}$, we define the collection of IVFs

$$[\widetilde{\mathcal{S}}(x)]_\alpha = \left[\underline{\mathcal{S}}_\alpha(x), \overline{\mathcal{S}}_\alpha(x)\right], \text{ where } \underline{\mathcal{S}}_\alpha, \overline{\mathcal{S}}_\alpha : \mathbb{R} \to \mathbb{R}.$$

Next, we will extend definitions 20 and 23, as discussed in [18], in the following manner:

Definition 4.1. *Let $x_0 \in \mathbb{R}$ and h be such that $x_0 + h \in \mathbb{R}$, then the gHD of $\widetilde{\mathcal{S}} : \mathbb{R} \to \mathcal{F}_{\mathcal{R}}$ at x_0 is*

$$\widetilde{\mathcal{S}}'(x_0) = \lim_{h\to 0} \frac{1}{h}[\widetilde{\mathcal{S}}(x_0 + h) \ominus_{\text{gH}} \widetilde{\mathcal{S}}(x_0)]. \tag{10}$$

If $\widetilde{\mathcal{S}}'(x_0) \in \mathcal{F}_{\mathcal{R}}$ satisfying Equation (10) exists, we say that $\widetilde{\mathcal{S}}$ is gHD at x_0. And also, the α-cut gHD of $\widetilde{\mathcal{S}} : \mathbb{R} \to \mathcal{F}_{\mathcal{R}}$ at x_0 is

$$[\widetilde{\mathcal{S}}'(x_0)]_\alpha = \lim_{h\to 0} \frac{1}{h}([\widetilde{\mathcal{S}}(x_0 + h)]_\alpha \ominus_{\text{gH}} [\widetilde{\mathcal{S}}(x_0)]_\alpha). \tag{11}$$

If $[\widetilde{\mathcal{S}}'(x_0)]_\alpha \in I_{\mathbb{R}}$ for all $\alpha \in [0,1]$, we say that $\widetilde{\mathcal{S}}$ is α-cut gHD at x_0.

Remark 4.2. *Let $\widetilde{\mathcal{S}}(x) = \widetilde{\mathcal{M}}\mathcal{T}(x)$, where $\mathcal{T}$ is a crisp polynomial function and $\widetilde{\mathcal{M}} \in \mathcal{F}_{\mathcal{R}}$. Therefore, we can easily infer the existence of the gHD, which is $\widetilde{\mathcal{S}}'(x) = \widetilde{\mathcal{M}}\mathcal{T}'(x)$.*

We extend the definition 26, as investigated in [18], in the following manner:

Definition 4.3. *Let $\tilde{\delta} : \mathbb{R} \to \mathcal{F}_{\mathcal{R}}$ and $x_0 \in \mathbb{R}$ with $\underline{\mathcal{S}}_\alpha$ and $\overline{\mathcal{S}}_\alpha$ both differentiable at x_0. For all $\alpha \in$ [0,1] we say that*

1). $\widetilde{\mathcal{S}}$ is (i)-gHD at x_0 if $[\widetilde{\mathcal{S}}'(x_0)]_\alpha = [\underline{\mathcal{S}}'_\alpha(x_0), \overline{\mathcal{S}}'_\alpha(x_0)]$.

2). $\widetilde{\mathcal{S}}$ is (ii)-gHD at x_0 if $[\widetilde{\mathcal{S}}'(x_0)]_\alpha = [\overline{\mathcal{S}}'_\alpha(x_0), \underline{\mathcal{S}}'_\alpha(x_0)]$.

The subsequent discovery provides a characterization of the two forms of gHD.

Theorem 4.4. *Let $\widetilde{\mathcal{S}}:\mathbb{R}\to\mathcal{F}_{\mathcal{R}}$, we assume that $\underline{\mathcal{S}}_\alpha$ and $\overline{\mathcal{S}}_\alpha$ are differentiable at $x_0\in\mathbb{R}$ with respect to α in the interval $[0,1]$. Subsequently*

a). $\widetilde{\mathcal{S}}$ is a type (i)-gHD if and only if $\underline{\mathcal{S}}'_\alpha(x_0)$ is nondecreasing, $\overline{\mathcal{S}}'_\alpha(x_0)$ is nonincreasing, as functions of α and $\underline{\mathcal{S}}'_1(x_0)\le\overline{\mathcal{S}}'_1(x_0)$.

b). $\widetilde{\mathcal{S}}$ is a type (ii)-gHD if and only if $\underline{\mathcal{S}}'_\alpha(x_0)$ is nonincreasing, $\overline{\mathcal{S}}'_\alpha(x_0)$ is nondecreasing, as functions of α and $\overline{\mathcal{S}}'_1(x_0)\le\underline{\mathcal{S}}'_1(x_0)$.

We will provide an example that demonstrates whether the gH-difference between two gHD FVFs is gHD or not.

Example 4.5. *Let's examine $\widetilde{\mathcal{S}},\widetilde{\mathcal{T}}:[-2,2]\in\mathbb{R}\to\mathcal{F}_{\mathcal{R}}$, where $\widetilde{\mathcal{S}}(x)=(0,1,2)\left(1-x^2\right)$ and $\widetilde{\mathcal{T}}(x)=(-3,-2,0,1)x$. The α-cuts of $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ are (see Figure 4-(a)-(b))*

$$[\widetilde{\mathcal{S}}(x)]_\alpha=\left[\alpha\left(1-x^2\right),(2-\alpha)\left(1-x^2\right)\right]\ \text{and}\ [\widetilde{\mathcal{T}}(x)]_\alpha=[(\alpha-3)x,(1-\alpha)x].$$

For gH-difference

$$\underline{\mathcal{S}}_\alpha-\underline{\mathcal{T}}_\alpha=\alpha\left(1-x^2\right)-(\alpha-3)x\le\overline{\mathcal{S}}_\alpha-\overline{\mathcal{T}}_\alpha=(2-\alpha)\left(1-x^2\right)-(1-\alpha)x, \tag{12}$$

in $[-2,-1.61)$ the gH -difference does not exist since Equation (12) does not satisfy Equation (9).-(a) or (9)-(b). But since Equation (12) satisfies equation (9)-(a) in $[-1.61,0]$ then gH-difference exists, so

$$[\widetilde{\mathcal{S}}]_\alpha\ominus_{\mathrm{gH}}[\widetilde{\mathcal{T}}]_\alpha=\left[\alpha\left(1-x^2\right)-(\alpha-3)x,(2-\alpha)\left(1-x^2\right)-(1-\alpha)x\right], \tag{13}$$

in interval $(0,0.61]$ the gH-difference does not exist since Equation (12) does not satisfy equation (9)-(a) or (9)-(b), but in $(0.61,2]$ the gH-difference exists as required by equation (9)-(b), so

$$[\widetilde{\mathcal{S}}]_\alpha\ominus_{\mathrm{gH}}[\widetilde{\mathcal{T}}]_\alpha=\left[(2-\alpha)\left(1-x^2\right)-(1-\alpha)x,\alpha\left(1-x^2\right)-(\alpha-3)x\right]. \tag{14}$$

See Figure 4-(c), and we can write the above results as follows:

$$[\widetilde{\mathcal{S}}]_\alpha\ominus_{\mathrm{gH}}[\widetilde{\mathcal{T}}]_\alpha=\begin{cases}\textit{does not exists} & \textit{when } x_0\in[-2,-1.61),\\ \textit{Equation (13)} & \textit{when } x_0\in[-1.61,0],\\ \textit{does not exists} & \textit{when } x_0\in(0,0.61],\\ \textit{Equation (14)} & \textit{when } x_0\in(0.61,2].\end{cases}$$

In the interval $[-2,0)$, both $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ satisfy condition (a) of Theorem 4.4 ; hence, they are (i)-gHD within this interval. In Equation (13), the endpoint functions are differentiable

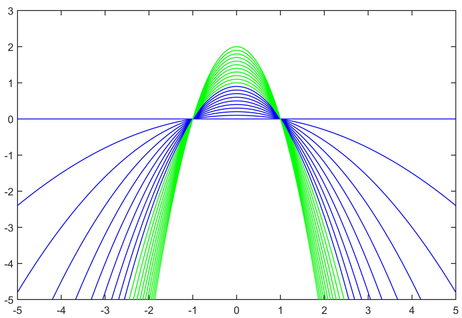
$$(\underline{\mathcal{S}\ominus_{\mathrm{gH}}\mathcal{T}})'_\alpha(x)=-2\alpha x-\alpha+3,\ (\overline{\mathcal{S}\ominus_{\mathrm{gH}}\mathcal{T}})'_\alpha(x)=-2(2-\alpha)x+\alpha-1. \tag{15}$$

Equation (15) satisfies condition (a) of Theorem 4.4 in the interval $(-1.61,-0.5)$; hence, $\widetilde{\mathcal{S}}\ominus_{\mathrm{gH}}\widetilde{\mathcal{T}}$ is (i)-gHD. But (15) satisfies condition (b) of Theorem 4.4 in $[-0.5,-0.1]$, so $\widetilde{\mathcal{S}}\ominus_{\mathrm{gH}}\widetilde{\mathcal{T}}$ is (ii)-gHD.

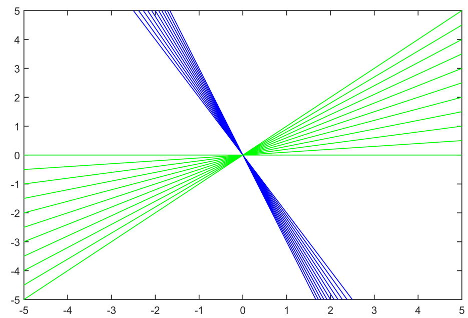
At $x_0 = 0, [\tilde{\mathcal{S}}'(0)]_\alpha = 0$, *then* $\widetilde{\mathcal{S}} \ominus_{gH} \widetilde{\mathcal{T}}$ *is (i)-gHD. In the interval* $(0, 2]$, $\widetilde{\mathcal{S}}$ *is (ii)-gHD and* $\widetilde{\mathcal{T}}$ *is (i)-gHD, then in Equation (14) the endpoint functions are evidently differentiable*

$$(\underline{\mathcal{S} \ominus_{\text{gH}} \mathcal{T}})'_\alpha(x) = -2(2-\alpha)x + \alpha - 1, \ \ (\overline{\mathcal{S} \ominus_{\text{gH}} \mathcal{T}})'_\alpha(x) = -2\alpha x - \alpha + 3. \tag{16}$$

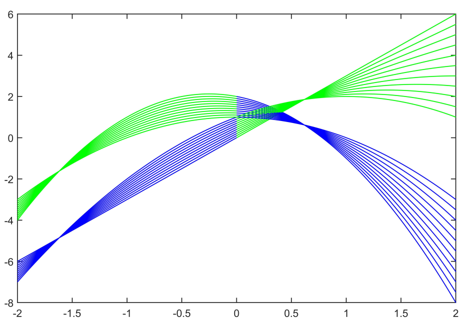
Equation (16) satisfies condition (a) of Theorem 4.4 in $[0.61, 2]$, *therefore* $\widetilde{\mathcal{S}} \ominus_{gH} \widetilde{\mathcal{T}}$ *is (i)-gHD, (see Figure 4-(d))*



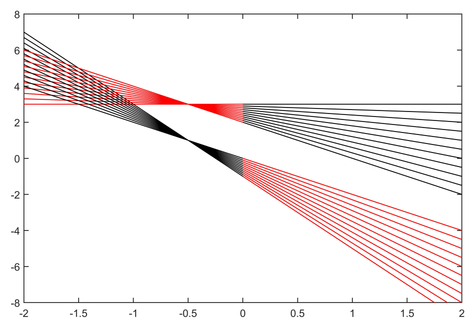
(a) $\underline{\mathcal{S}}_\alpha$ blues and $\overline{\mathcal{S}}_\alpha$ greens



(b) $\underline{\mathcal{T}}_\alpha$ blues and $\overline{\mathcal{T}}_\alpha$ greens



(c) $(\underline{\mathcal{S} + \mathcal{T}})_\alpha$ blues and $(\overline{\mathcal{S} + \mathcal{T}})_\alpha$ greens



(d) $(\underline{\mathcal{S} + \mathcal{T}})'_\alpha$ reds and $(\overline{\mathcal{S} + \mathcal{T}})'_\alpha$ blacks

Figure 4: The level sets of FVFs in Example 4.5 for $\alpha = 0, 0.1, ..., 1$.

Next, we prove that the summation of two gHD FVFs is gHD, when they are in the same type of gHD.

Theorem 4.6. *If* $\widetilde{\mathcal{S}}$ *and* $\widetilde{\mathcal{T}}$ *are both type (i)-gHD or both type (ii)-gHD, then*

1). $(\widetilde{\mathcal{S}} + \widetilde{\mathcal{T}})'_{(i)-\text{gHD}} = \widetilde{\mathcal{S}}'_{(i)-\text{gHD}} + \widetilde{\mathcal{T}}'_{(i)-\text{gHD}}$

2). $(\widetilde{\mathcal{S}} + \widetilde{\mathcal{T}})'_{(ii)-\text{gHD}} = \widetilde{\mathcal{S}}'_{(ii)-\text{gHD}} + \widetilde{\mathcal{T}}'_{(ii)-\text{gHD}}$

Proof. (1). Let $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ are both (i)-gHD; for every α in $[0, 1]$ then $[\tilde{\mathcal{S}}']_\alpha = [(\underline{\mathcal{S}}_\alpha)', (\overline{\mathcal{S}}_\alpha)']$ and $[\widetilde{\mathcal{T}}']_\alpha = [(\underline{\mathcal{T}}_\alpha)', (\overline{\mathcal{T}}_\alpha)']$. Using Equation (1) yields that

$$\begin{aligned}[(\widetilde{\mathcal{S}}+\widetilde{\mathcal{T}})']_\alpha &= [(\underline{\mathcal{S}}_\alpha+\underline{\mathcal{T}}_\alpha),(\overline{\mathcal{S}}_\alpha+\overline{\mathcal{T}}_\alpha)]' \\ &= [(\underline{\mathcal{S}}_\alpha+\underline{\mathcal{T}}_\alpha)',(\overline{\mathcal{S}}_\alpha+\overline{\mathcal{T}}_\alpha)'] \\ &= [(\underline{\mathcal{S}}_\alpha)'+(\underline{\mathcal{T}}_\alpha)',(\overline{\mathcal{S}}_\alpha)'+(\overline{\mathcal{T}}_\alpha)'] \\ &= [(\underline{\mathcal{S}}_\alpha)',(\bar{f}_\alpha)']+[(\underline{\mathcal{T}}_\alpha)',(\overline{\mathcal{T}}_\alpha)'] \\ &= [\widetilde{\mathcal{S}}']_\alpha+[\widetilde{\mathcal{T}}']_\alpha.\end{aligned}$$

In the case of (2), let $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ are both (ii)-gHD; for all $\alpha\in[0,1]$ then $[\widetilde{\mathcal{S}}']_\alpha=[(\overline{\mathcal{S}}_\alpha)',(\underline{\mathcal{S}}_\alpha)']$ and $[\widetilde{\mathcal{T}}']_\alpha=[(\overline{\mathcal{T}}_\alpha)',(\underline{\mathcal{T}}_\alpha)']$. Using Equation (1) yields that

$$\begin{aligned}[(\widetilde{\mathcal{S}}+\widetilde{\mathcal{T}})']_\alpha &= [(\underline{\mathcal{S}}_\alpha+\underline{\mathcal{T}}_\alpha),(\overline{\mathcal{S}}_\alpha+\overline{\mathcal{T}}_\alpha)]' \\ &= [(\overline{\mathcal{S}}_\alpha+\overline{\mathcal{T}}_\alpha)',(\underline{\mathcal{S}}_\alpha+\underline{\mathcal{T}}_\alpha)'] \\ &= [(\overline{\mathcal{S}}_\alpha)'+(\overline{\mathcal{T}}_\alpha)',(\underline{\mathcal{S}}_\alpha)'+(\underline{\mathcal{T}}_\alpha)'] \\ &= [(\overline{\mathcal{S}}_\alpha)',(\underline{\mathcal{S}}_\alpha)']+[(\overline{\mathcal{T}}_\alpha)',(\underline{\mathcal{T}}_\alpha)'] \\ &= [\widetilde{\mathcal{S}}']_\alpha+[\widetilde{\mathcal{T}}']_\alpha.\end{aligned}$$

□

Example 4.7. *We consider $\widetilde{\mathcal{S}},\widetilde{\mathcal{T}}:\mathbb{R}\to\mathcal{F}_\mathcal{R}$, where $\widetilde{\mathcal{S}}(x)=(0,1,2,3)x^2$ and $\widetilde{\mathcal{T}}(x)=(-1,0,1,4)x$. The α-cuts of $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ are*

$$[\widetilde{\mathcal{S}}(x)]_\alpha=\left[\alpha x^2,(3-\alpha)x^2\right] \text{ and } [\widetilde{\mathcal{T}}(x)]_\alpha=[(\alpha-1)x,(4-3\alpha)x],$$

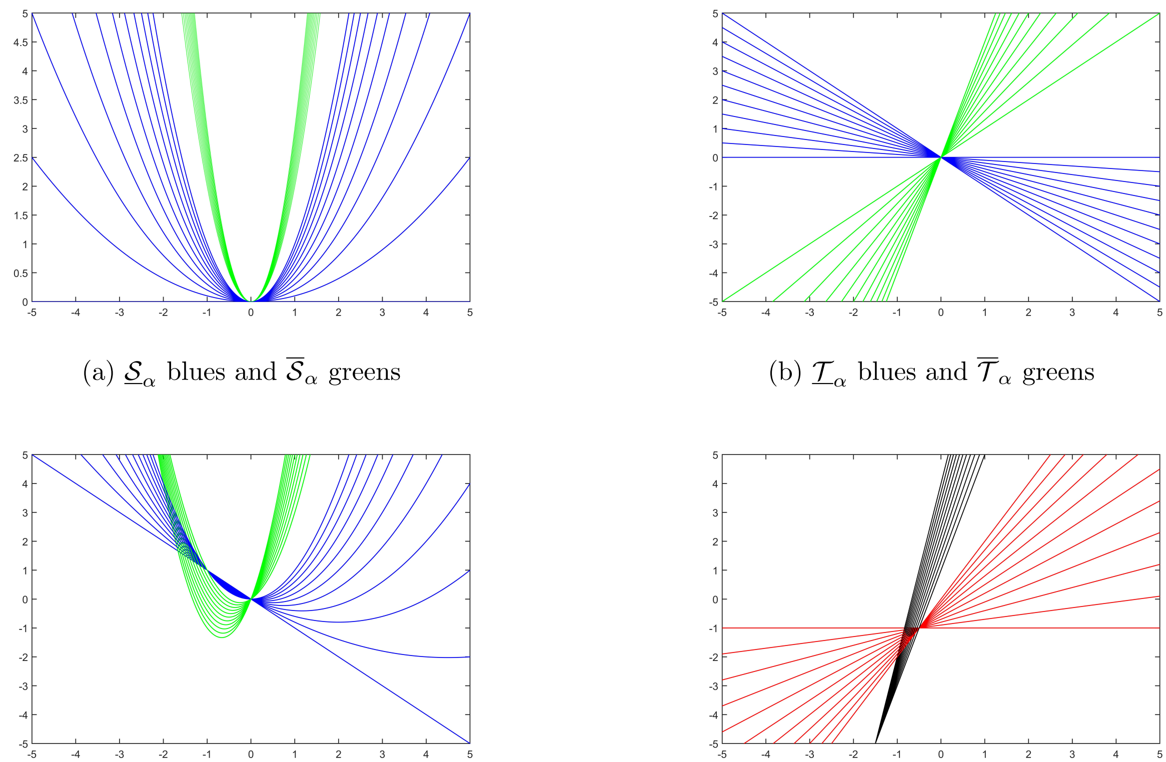
the endpoint functions are differentiable at any $x_0\in\mathbb{R}$ (see Figure 5-(a)-(b)). Now, the endpoint functions of $\widetilde{\mathcal{S}}+\widetilde{\mathcal{T}}$ are (see Figure 5-(c))

$$(\underline{\mathcal{S}+\mathcal{T}})_\alpha(x)=\alpha x^2+(\alpha-1)x,\ (\overline{\mathcal{S}+\mathcal{T}})_\alpha(x)=(3-\alpha)x^2+(4-3\alpha)x.$$

The endpoint functions are evidently differentiable, (see Figure 5-(d))

$$(\underline{\mathcal{S}+\mathcal{T}})'_\alpha(x)=2\alpha x+\alpha-1,\ (\overline{\mathcal{S}+\mathcal{T}})'_\alpha(x)=2(3-\alpha)x+4-3\alpha. \quad (17)$$

If $x_0<0,\widetilde{\mathcal{S}}$ is (ii)-gHD and $\widetilde{\mathcal{T}}$ is (i)-gHD. Equation (17) satisfies condition (b) of Theorem 4.4 in $(-\infty,-1.5]$; therefore, $\widetilde{\mathcal{S}}+\widetilde{\mathcal{T}}$ is (ii)-gHD. However, Equation (17) fails to satisfy either (a) or (b) of Theorem 4.4 in $(-1.5,-0.5)$, therefore $\widetilde{\mathcal{S}}+\widetilde{\mathcal{T}}$ is not gHD in this interval. Similarly, $\widetilde{\mathcal{S}}+\widetilde{\mathcal{T}}$ is (i)-gHD in $[-0.5,-0.1]$. If $x_0=0$, then $[\widetilde{\mathcal{S}}'(0)]_\alpha=0$ and $\widetilde{\mathcal{T}}$ is (i)-gHD; therefore $\widetilde{\mathcal{S}}+\widetilde{\mathcal{T}}$ is (i)-gHD. If $x_0>0$, we have that $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ are both (i)-gHD. Thus, Equation (17) satisfies condition (a) of Theorem 4.4 at any $x_0>0$. Therefore $\widetilde{\mathcal{S}}+\widetilde{\mathcal{T}}$ is (i)-gHD.



(a) $\underline{\mathcal{S}}_\alpha$ blues and $\overline{\mathcal{S}}_\alpha$ greens

(b) $\underline{\mathcal{T}}_\alpha$ blues and $\overline{\mathcal{T}}_\alpha$ greens

(c) $(\underline{\mathcal{S}+\mathcal{T}})_\alpha$ blues and $(\overline{\mathcal{S}+\mathcal{T}})_\alpha$ greens

(d) $(\underline{\mathcal{S}+\mathcal{T}})'_\alpha$ reds and $(\overline{\mathcal{S}+\mathcal{T}})'_\alpha$ blacks

Figure 5: The level sets of FVFs in Example 4.7 for $\alpha = 0, 0.1, ..., 1$.

Remark 4.8. *In the example above, we have seen that when we have two gHD FVFs in different types of gHD, the summation of these two gHD FVFs is not necessarily gHD or may have different results in different intervals.*

Next, we prove that the subtraction of two gHD FVFs become gHD, when they are in the same types of gHD.

Theorem 4.9. *If $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ are both type (i)-gHD or both type (ii)-gHD, then*

1). $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{T}})'_{(i)-\text{gHD}} = \widetilde{\mathcal{S}}'_{(i)-\text{gHD}} - \widetilde{\mathcal{T}}'_{(i)-\text{gHD}}$

2). $(\widetilde{\mathcal{S}} - \widetilde{\mathcal{T}})'_{(ii)-\text{gHD}} = \widetilde{\mathcal{S}}'_{(ii)-\text{gHD}} - \widetilde{\mathcal{T}}'_{(ii)-\text{gHD}}$

Proof. Examine case (1) and assume that $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ are both (i)-gHD; then, for every α in $[0,1]$, we have $[\widetilde{\mathcal{S}}']_\alpha = [(\underline{\mathcal{S}}_\alpha)', (\overline{\mathcal{S}}_\alpha)']$ and $[\widetilde{\mathcal{T}}']_\alpha = [(\mathcal{T}_\alpha)', (\overline{\mathcal{T}}_\alpha)']$. Using Equation (2) yields that

$$\begin{aligned}[(\widetilde{\mathcal{S}} - \widetilde{\mathcal{T}})']_\alpha &= [(\underline{\mathcal{S}}_\alpha - \overline{\mathcal{T}}_\alpha), (\overline{\mathcal{S}}_\alpha - \underline{\mathcal{T}}_\alpha)]' \\ &= [(\underline{\mathcal{S}}_\alpha - \overline{\mathcal{T}}_\alpha)', (\overline{\mathcal{S}}_\alpha - \underline{\mathcal{T}}_\alpha)'] \\ &= [(\underline{\mathcal{S}}_\alpha)' - (\overline{\mathcal{T}}_\alpha)', (\overline{\mathcal{S}}_\alpha)' - (\overline{\mathcal{T}}_\alpha)'] \\ &= [(\underline{\mathcal{S}}_\alpha)', (\overline{\mathcal{S}}_\alpha)'] - [(\underline{\mathcal{T}}_\alpha)', (\overline{\mathcal{T}}_\alpha)'] \\ &= [\widetilde{\mathcal{S}}']_\alpha - [\widetilde{\mathcal{T}}']_\alpha.\end{aligned}$$

We demonstrate part (2) using the same rationale as the preceding proof for part (1). □

Example 4.10. *We consider $\widetilde{\mathcal{S}}, \widetilde{\mathcal{T}} : \mathbb{R} \to \mathcal{F}_\mathcal{R}$, where $\widetilde{\mathcal{S}}(x) = (0,1,2)x^3$ and $\widetilde{\mathcal{T}}(x) = (1,2,4)x$. The α-cuts of $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ are*

$$[\widetilde{\mathcal{S}}(x)]_\alpha = [\alpha x^3, (2-\alpha)x^3] \text{ and } [\widetilde{\mathcal{T}}(x)]_\alpha = [(1+\alpha)x, (4-2\alpha)x].$$

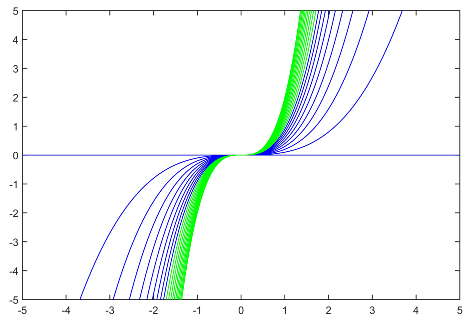
The endpoint functions are differentiable (see Figure 6-(a)-(b).). Now, the components of $[\widetilde{\mathcal{S}} - \widetilde{\mathcal{T}}]_\alpha$ are (see Figure 6-(c))

$$(\underline{\mathcal{S} - \mathcal{T}})_\alpha(x) = \alpha x^3 - (4-2\alpha)x, \ (\overline{\mathcal{S} - \mathcal{T}})_\alpha(x) = (2-\alpha)x^3 - (1+\alpha)x.$$

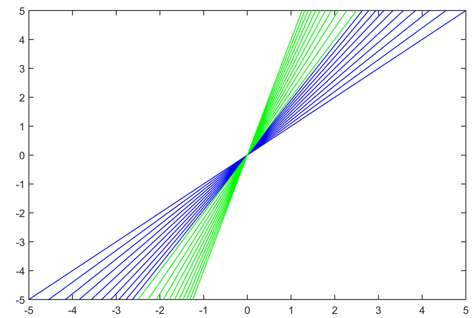
The endpoint functions are differentiable (see Figure 6-(d))

$$(\underline{\mathcal{S} - \mathcal{T}})'_\alpha(x) = 3\alpha x^2 + 2\alpha - 4, \ (\overline{\mathcal{S} - \mathcal{T}})'_\alpha(x) = 3(2-\alpha)x^2 - \alpha - 1. \tag{18}$$

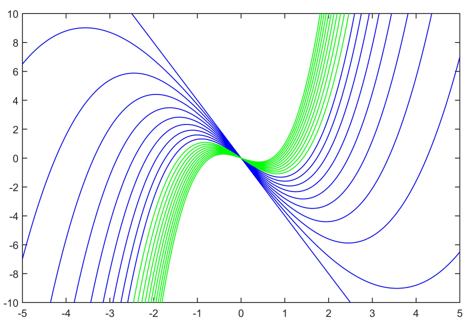
Now, for $x_0 > 0$ and $x_0 < 0$ we have that $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ are both (i)-gHD. Thus, Equation (18) satisfies condition (a) of Theorem 4.4. Therefore $\widetilde{\mathcal{S}} - \widetilde{\mathcal{T}}$ is (i)-gHD. If $x_0 = 0$, then $[\widetilde{\mathcal{S}}'(0)]_\alpha = 0$ and $\widetilde{\mathcal{T}}$ is (i)-gHD; therefore $\widetilde{\mathcal{S}} - \widetilde{\mathcal{T}}$ is (i)-gHD.



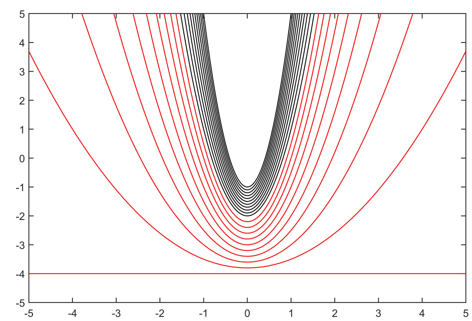
(a) $\underline{\mathcal{S}}_\alpha$ blues and $\overline{\mathcal{S}}_\alpha$ greens



(b) $\underline{\mathcal{T}}_\alpha$ blues and $\overline{\mathcal{T}}_\alpha$ greens



(c) $(\underline{\mathcal{S}-\mathcal{T}})_\alpha$ blues and $(\overline{\mathcal{S}-\mathcal{T}})_\alpha$ greens



(d) $(\underline{\mathcal{S}-\mathcal{T}})'_\alpha$ reds and $(\overline{\mathcal{S}-\mathcal{T}})'_\alpha$ blacks

Figure 6: The level sets of FVFs in Example 4.10 for $\alpha = 0, 0.1, ..., 1$.

Remark 4.11. *If we apply $\widetilde{\mathcal{S}} - \widetilde{\mathcal{T}}$ to Example 4.7, then we will have the same cases of the summation that the subtraction of two gHD FVFs might not be gHD or may have different results in different intervals when they are in different types of gHD.*

For any $\lambda \in \mathbb{R}$, we will now examine the gHD of $\lambda\widetilde{\mathcal{S}}$.

Theorem 4.12. *If $\widetilde{\mathcal{S}}$ is type (i)-gHD or type (ii)-gHD, then*

1). $(\lambda\widetilde{\mathcal{S}})'_{(i)-\text{gHD}} = \lambda(\widetilde{\mathcal{S}})'_{(i)-\text{gHD}}$

2). $(\lambda\widetilde{\mathcal{S}})'_{(ii)-\text{gHD}} = \lambda(\widetilde{\mathcal{S}})'_{(ii)-\text{gHD}}$

Proof. (1). Let $\widetilde{\mathcal{S}}$ is (i)-gHD, $\forall\alpha \in [0,1]$ then $[\widetilde{\mathcal{S}}']_\alpha = [(\underline{\mathcal{S}}_\alpha)', (\overline{\mathcal{S}}_\alpha)']$. By using Equation (5), first suppose that $\lambda > 0$ then

$$\begin{aligned}[\lambda\widetilde{\mathcal{S}}']_\alpha &= [\lambda(\underline{\mathcal{S}}_\alpha)', \lambda(\overline{\mathcal{S}}_\alpha)'] \\ &= \lambda[(\underline{\mathcal{S}}_\alpha)', (\overline{\mathcal{S}}_\alpha)'] \\ &= \lambda[\widetilde{\mathcal{S}}']_\alpha.\end{aligned}$$

Now, assume that $\lambda < 0$ then

$$\begin{aligned}[\lambda\widetilde{\mathcal{S}}']_\alpha &= [\lambda(\overline{\mathcal{S}}_\alpha)', \lambda(\underline{\mathcal{S}}_\alpha)'] \\ &= \lambda[(\underline{\mathcal{S}}_\alpha)', (\overline{\mathcal{S}}_\alpha)'] \\ &= \lambda[\widetilde{\mathcal{S}}']_\alpha.\end{aligned}$$

Ultimately, if $\lambda = 0$, the outcome is self-evident. We demonstrate part (2) using the same rationale as the preceding proof for part (1). □

5 Conclusion

In this paper, we have studied some arithmetic operations on real compact intervals and on FNs through their α-cuts. We provided conditions for the existence of the gH-difference between FNs and extended these ideas to FVFs. We have clarified several characteristics of the algebra related to gHD of FVFs. Specifically, we have explained how to calculate the gHD of the gH-difference, summation and subtraction of two gHD FVFs. Furthermore, we have established a formula for deriving the gHD of FVFs generated by the multiplication of a real number with a gHD FVF. We concluded that the gH-difference, summation and subtraction of two gHD FVFs is not always gHD. Therefore, we proposed special conditions to show that these operations on the gHD of FVFs are gHD. In future research, we will identify new characteristics of the algebra of gHD FVFs, including a formula to create a gHD FVF by multiplying and dividing two gHD FVFs. Additionally, we plan to apply the same concepts mentioned earlier to other types of fuzzy functions.

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