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Research Article

Existence and Uniqueness of Solutions for Riemann-Liouville Fractional High-Order Multi-Point Boundary Value Problems

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Abstract

Abstract: In this paper, we investigate the existence and uniqueness of solutions for a nonlinear fractional differential equation involving the Riemann-Liouville fractional derivative of order $\vartheta \in (m-1, m]$. We construct the Green's function for the corresponding linear boundary value problem and analyze its properties. By employing the Banach contraction mapping principle, we establish sufficient conditions for the existence of a unique solution. A specific emphasis is placed on the structural differences between Riemann-Liouville and Caputo frameworks, particularly regarding the general solutions and the behavior of the Green's function integral bounds.
Keywords: High-order fractional differential equation; Riemann-Liouville fractional derivative; Boundary value problem; Existence and uniqueness; Fixed point theorem.
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1 Introduction

Fractional calculus has garnered significant attention due to its extensive applications in modeling complex systems with memory and hereditary properties [1, 2, 3, 4, 5, 6, 7, 8, 9]. Among the various definitions of fractional derivatives, the Riemann-Liouville and Caputo derivatives are the most prominent. While the Caputo derivative is often preferred in applied sciences because it allows for traditional integer-order initial conditions, the Riemann-Liouville derivative poses unique mathematical challenges due to the singular nature of its solutions near the origin.

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The theory of fixed points is an indispensable and powerful mathematical tool for investigating the existence and uniqueness of solutions to boundary value problems. This methodology not only rigorously confirms the existence of solutions but also provides a framework for deriving approximate solutions. The profound importance of this approach in the context of boundary value problems is extensively highlighted in key references such such as [10, 11, 12, 13, 14].

In this work, we consider the following boundary value problem involving the Riemann-Liouville fractional derivative:

$$D_{0+}^{\vartheta} w(t) + f(t, w(t)) = 0, \quad t \in [0, 1],$$

subject to the boundary conditions:

$$w(0) = w'(0) = \cdots = w^{(m-2)}(0) = 0, \quad w(1) = 0,$$

where $m - 1 < \vartheta \leq m$ for $m \geq 2$, D_{0+}^{ϑ} is the standard Riemann-Liouville fractional derivative, and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

2 Preliminaries

We first recall some fundamental definitions from fractional calculus.

Definition 2.1. The Riemann-Liouville fractional integral of order $\vartheta > 0$ of a function $h \in L^1([0, 1])$ is defined as:

$$I_{0+}^{\vartheta} h(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t - s)^{\vartheta - 1} h(s) ds.$$

Definition 2.2. The Riemann-Liouville fractional derivative of order $\vartheta > 0$ of a continuous function h is defined by:

$$D_{0+}^{\vartheta} h(t) = \frac{1}{\Gamma(m - \vartheta)} \frac{d^m}{dt^m} \int_0^t (t - s)^{m - \vartheta - 1} h(s) ds,$$

where $m = [\vartheta] + 1$.

Lemma 2.3. *Let $\vartheta > 0$. The general solution to the homogeneous fractional differential equation $D_{0+}^{\vartheta} w(t) = 0$ is given by:*

$$w(t) = c_1 t^{\vartheta - 1} + c_2 t^{\vartheta - 2} + \cdots + c_m t^{\vartheta - m},$$

where $c_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$.

3 Green's Function and its Properties

Lemma 3.1. *Let $h \in C([0, 1])$. The unique solution of the linear boundary value problem*

$$D_{0+}^{\vartheta} w(t) + h(t) = 0, \quad 0 \leq t \leq 1,$$

$$w(0) = w'(0) = \cdots = w^{(m-2)}(0) = 0, \quad w(1) = 0,$$

is given by

$$w(t) = \int_0^1 R(t, s) h(s) ds,$$

where the Green's function $R(t, s)$ *is defined as:*

$$R(t,s) = \begin{cases} \frac{t^{\vartheta-1}(1-s)^{\vartheta-1}-(t-s)^{\vartheta-1}}{\Gamma(\vartheta)}, & 0 \le s \le t \le 1; \\ \frac{t^{\vartheta-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)}, & 0 \le t \le s \le 1. \end{cases}$$

Proof. By applying the fractional integral I_{0+}^{ϑ} to the equation $D_{0+}^{\vartheta} w(t) = -h(t)$ and using Lemma 2.3, the general solution is:

$$w(t) = c_1 t^{\vartheta-1} + c_2 t^{\vartheta-2} + \cdots + c_m t^{\vartheta-m} - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s) ds.$$

Because $m - 1 < \vartheta \le m$, the terms $t^{\vartheta-2}, \ldots, t^{\vartheta-m}$ are singular at $t = 0$. To satisfy the boundary conditions $w(0) = w'(0) = \cdots = w^{(m-2)}(0) = 0$ and avoid undefined values, we must set $c_2 = c_3 = \cdots = c_m = 0$. This yields:

$$w(t) = c_1 t^{\vartheta-1} - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s) ds.$$

Applying the condition $w(1) = 0$, we find:

$$c_1 (1)^{\vartheta-1} - \frac{1}{\Gamma(\vartheta)} \int_0^1 (1-s)^{\vartheta-1} h(s) ds = 0 \implies c_1 = \frac{1}{\Gamma(\vartheta)} \int_0^1 (1-s)^{\vartheta-1} h(s) ds.$$

Substituting c_1 back into the equation for $w(t)$ provides:

$$w(t) = \frac{t^{\vartheta-1}}{\Gamma(\vartheta)} \int_0^1 (1-s)^{\vartheta-1} h(s) ds - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s) ds.$$

Rearranging the integrals yields the Green's function $R(t, s)$ as defined. ■

Remark 3.2. It is crucial to note that the Riemann-Liouville Green's function involves the fractional term $t^{\vartheta-1}$, in contrast to the Caputo framework which yields a polynomial term t^{m-1}.

Lemma 3.3. *The Green's function* $R(t, s)$ *satisfies the following integral bound for all* $t \in [0, 1]$*:*

$$\int_0^1 |R(t,s)| ds \le \frac{2}{\Gamma(\vartheta+1)}.$$

Proof. Direct integration yields:

$$\begin{aligned} \int_0^1 |R(t,s)| ds &= \int_0^t \frac{t^{\vartheta-1}(1-s)^{\vartheta-1} - (t-s)^{\vartheta-1}}{\Gamma(\vartheta)} ds + \int_t^1 \frac{t^{\vartheta-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} ds \\ &\le \int_0^1 \frac{t^{\vartheta-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} ds + \int_0^t \frac{(t-s)^{\vartheta-1}}{\Gamma(\vartheta)} ds \\ &= \frac{t^{\vartheta-1}}{\Gamma(\vartheta+1)} + \frac{t^{\vartheta}}{\Gamma(\vartheta+1)}. \end{aligned}$$

Because $t \in [0, 1]$ and $\vartheta - 1 > 0$, we have $t^{\vartheta-1} \le 1$ and $t^{\vartheta} \le 1$. Consequently:

$$\frac{t^{\vartheta-1} + t^{\vartheta}}{\Gamma(\vartheta+1)} \le \frac{1+1}{\Gamma(\vartheta+1)} = \frac{2}{\Gamma(\vartheta+1)}.$$

This establishes the bound. Although the intermediate step involves $t^{\vartheta-1}$ rather than t^{m-1}, the maximal bound on $[0, 1]$ coincides with the Caputo case due to the properties of fractional powers on the unit interval. ■

4 Main Results

Let $X = C([0,1],\mathbb{R})$ be the Banach space of all continuous functions with the norm $\|w\| = \max_{t\in[0,1]} |w(t)|$. We define the operator $T : X \to X$ by:

$$(Tw)(t) = \int_0^1 R(t,s)f(s,w(s))ds.$$

Theorem 4.1. *Assume that $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and there exists a constant $L > 0$ such that:*

$$|f(t,u) - f(t,v)| \le L|u-v|, \quad \textit{for all } t \in [0,1], \textit{ and } u,v \in \mathbb{R}.$$

If $\frac{2L}{\Gamma(\vartheta+1)} < 1$, then the boundary value problem has a unique solution on $[0,1]$.

Proof. For any $u, v \in X$ and $t \in [0,1]$, using Lemma 3.3 we have:

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| &\le \int_0^1 R(t,s)|f(s,u(s)) - f(s,v(s))|ds \\
&\le L\int_0^1 R(t,s)|u(s) - v(s)|ds \\
&\le L\|u-v\|\int_0^1 |R(t,s)|ds \\
&\le \frac{2L}{\Gamma(\vartheta+1)}\|u-v\|.
\end{aligned}$$

Taking the supremum over $t \in [0,1]$ yields:

$$\|Tu - Tv\| \le \frac{2L}{\Gamma(\vartheta+1)}\|u-v\|.$$

Since $\frac{2L}{\Gamma(\vartheta+1)} < 1$, the operator T is a contraction. By the Banach fixed-point theorem, T has a unique fixed point in X, which is the unique solution to the problem. ∎

5 Conclusion

We have established the existence and uniqueness of solutions for a boundary value problem involving the Riemann-Liouville fractional derivative. By carefully deriving the corresponding Green's function, we highlighted the foundational differences between the Riemann-Liouville and Caputo solutions. We demonstrated that despite the structural differences in the Green's functions, the topological arguments for existence and uniqueness yield an identical constraint condition due to the bounding behavior of fractional polynomials on the unit interval.

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