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Research Article

Approximate Multi-Quintic-Sextic Mappings

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Abstract

Abstract: In the present paper, we introduce a multi-quintic-sextic mapping as a system of functional equations taken from quintic and sextic functional equations. We describe the structure of such mappings and characterize them. In other words, we show that each multi-quintic-sextic mapping can be unified as a single equation. In the special cases, such mappings are multi-quintic and multi-sextic. Furthermore, by a classical direct (Hyers) method of stability, we establish the stability of multi-quintic-sextic mappings in the setting of Banach spaces.

Keywords: Multi-quintic mapping, Multi-sextic mapping, Multi-quintic-sextic mapping, Stability.
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1 Introduction

In 1940, for the first time, Ulam [1] proposed a stability problem, asking whether every approximately homomorphic mapping between groups must be near to a homomorphism. One year later, Hyers [2] provided an affirmative answer to the Ulam's question in Banach spaces by applying the so-called direct method, thereby establishing the foundation of the stability theory of functional equations. In 1950, Aoki [3] and in 1978, Rassias [4] significantly extended Hyers' result by using the direct method and weakening the condition on the bound of the Cauchy difference; the generalized outcome of the former results can be found in [5]. After that, the stability of Ulam-Hyers-Rassias for various functional equations on miscellaneous spaces have been studied by numerous authors which are available in literature. Furthermore, in recent years, the characterization and stability problem for multiple variable mappings with their solutions have been explored by the authors;

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multi-additive [6, 7], multi-quadratic [8, 9, 10, 11], multi-cubic [12, 13] and multi-quartic mappings [14]. For multi-additive-quadratic mapping, we refer to [15].

Regarding quintic functional equations, Xu et al. [16] introduced the quintic functional equation

$$\begin{aligned} f(3x+y)-5f(2x+y)+&f(2x-y)+10f(x+y)-5f(x-y)\\ &=10f(y)+f(3x)-3f(2x)-27f(x) \end{aligned}\tag{1}$$

and obtained its general solutions. Moreover, they established the stability of (1) in the setting of quasi-β-normed spaces. Next, Cho et al. [17] presented the quintic functional equation

$$\begin{aligned} 2Q(2x+y)+2Q(2x-y)+Q(x+2y)+&Q(x-2y)\\ &=20[Q(x+y)+Q(x-y)]+90Q(x), \end{aligned}\tag{2}$$

and then explored the Hyers-Ulam stability of (2) in quasi-β-normed spaces setting.

In [16], for the first time, Xu et al. found the general solution of the sextic functional equation

$$\begin{aligned} f(x+3y)-6f(x+2y)+&15f(x+y)-20f(x)+15f(x-y)\\ &=6f(x-2y)-f(x-3y)+720f(y). \end{aligned}\tag{3}$$

They also studied the Ulam stability problem for (3) in quasi-β-normed spaces via a fixed point method. A new form of the sextic functional equation is

$$\begin{aligned} &S(2x+y)+S(2x-y)+S(x+2y)+S(x-2y)\\ &\quad=20[S(x+y)+S(x-y)]+90[S(x)+S(y)], \end{aligned}\tag{4}$$

which was introduced by Ravi et al. [18].

Motivated by the quintic functional equation (2), Bodaghi et al. [19] introduced multi-quintic mappings those fulfill (2) in each component and included a characterization of such mappings. Furthermore, they established the Hyers-Ulam stability for multi-quintic mappings by applying the fixed point method in normed spaces. Moreover, in [20], the first author defined the multi-sextic mappings (taken from (4)). He also showed that every multi-sextic mapping can be shown a single functional equation and vice versa (under some extra conditions). Next, he proved the Hyers-Ulam and Găvruţa stability for the multi-sextic mappings in non-Archimedean normed and quasi-β-normed spaces.

Following [20] and [19], in the current work, for two linear spaces $\mathbb{V}$ and $\mathbb{W}$, we define a multi-quintic-sextic mapping $\Gamma:\mathbb{V}^n\longrightarrow\mathbb{W}$ as a system of functional equations which is quintic in each of some k ($k<n$) variables and sextic in each of the other variables. We represent each multi-quintic-sextic mapping as an unified single equation. Moreover, by a classical direct (Hyers) method of stability, we establish the stability of multi-quintic-sextic mappings in the setting of Banach spaces.

2 Representation of multi-quintic-sextic mappings

We begin this section with the definition of multi-quintic-sextic mappings. Throughout this section, $\mathbb{V}$ and $\mathbb{W}$ are linear spaces and $k\in\{0,1,\dots,n\}$.

Definition 2.1. A mapping $\Phi : \mathbb{V}^n \longrightarrow \mathbb{W}$ is called k-quintic and $n-k$-sextic or multi-quintic-sextic if it satisfies the system of the equations as follows:

$$\begin{aligned}
&2[\Gamma(\omega_1,\dots,\omega_{i-1},2\omega_i+\omega_i',\omega_{i+1},\dots,\omega_n)+\Gamma(\omega_1,\dots,\omega_{i-1},2\omega_i-\omega_i',\omega_{i+1},\dots,\omega_n)]\\
&\quad+\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i+2\omega_i',\omega_{i+1},\dots,\omega_n)+\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i-2\omega_i',\omega_{i+1},\dots,\omega_n)\\
&=20[\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i+\omega_i',\omega_{i+1},\dots,\omega_n)+\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i-\omega_i',\omega_{i+1},\dots,\omega_n)]\\
&\quad+90\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i,\omega_{i+1},\dots,\omega_n)\qquad(i\in\{1,\dots,k\}),\\
&\Gamma(\omega_1,\dots,\omega_{i-1},2\omega_i+\omega_i',\omega_{i+1},\dots,\omega_n)+\Gamma(\omega_1,\dots,\omega_{i-1},2\omega_i-\omega_i',\omega_{i+1},\dots,\omega_n)\\
&\quad+\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i+2\omega_i',\omega_{i+1},\dots,\omega_n)+\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i-2\omega_i',\omega_{i+1},\dots,\omega_n)\\
&=20[\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i+\omega_i',\omega_{i+1},\dots,\omega_n)+\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i-\omega_i',\omega_{i+1},\dots,\omega_n)]\\
&\quad+90[\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i,\omega_{i+1},\dots,\omega_n)+\Gamma(\omega_1,\dots,\omega_{i-1},\omega_i',\omega_{i+1},\dots,\omega_n)],
\end{aligned}$$

for $i \in \{k+1,\dots,n\}$.

Note that in the definition above, Φ is quintic in each of some k variables (see equation (2)) and is sextic in each of the other variables (see equation (4)). In this note, we suppose for simplicity that Φ is quintic in each of the first k variables and is sextic in each of the other variables but one can obtain analogous results without this assumption. Let us note that for $k=n$ and $k=0$, the above definition leads to the multi-quintic and multi-sextic mappings, respectively (see [20] and [19]). For example, the function $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined through $\Phi(a_1,\dots,a_n)=\prod_{i=1}^k\prod_{j=k+1}^n a_i^5 a_j^6$ is a multi-quintic-sextic function.

We denote $\mathbb{V}^n$ sometimes by $\mathbb{V}^k \times \mathbb{V}^{n-k}$. For $i \in \{1,2\}$, we put $v_i^{[k]}=(v_{i1},\dots,v_{ik}) \in \mathbb{V}^k$ and $v_i^{[n-k]}=(v_{i,k+1}\dots,v_{in}) \in \mathbb{V}^{n-k}$. Let $v_1^{[k]}, v_2^{[k]} \in \mathbb{V}^k$. Assume that $t_1,t_2 \in \mathbb{N}_0=\mathbb{N}\bigcup\{0\}$ with $0 \leq t_1,t_2 \leq k$. Set $\mathcal{V}^k=\{\mathfrak{V}_k=(V_1,\dots,V_k)|\ V_j \in \{v_{1j}\pm v_{2j}, v_{1j}\pm 2v_{2j}, v_{1j}\}$, where $j \in \{1,\dots,k\}$. Consider the subset $\mathcal{V}^k_{(t_1,t_2)}$ of $\mathcal{V}^k$ as follows:

$$\mathcal{V}^k_{(t_1,t_2)} := \left\{\mathfrak{V}_k \in \mathcal{V}^k|\ \mathrm{Card}\{V_j : V_j = x_{1j}\} = t_1,\ \mathrm{Card}\{V_j : V_j = v_{1j}\pm v_{2j}\} = t_2\right\},$$

where $\mathrm{Card}\Omega$ is the cardinality of a set Ω. Moreover, we put

$$\mathcal{V}^{n-k}=\{\mathfrak{V}_{n-k}=(V_{k+1},\dots,V_n)|\ V_j \in \{v_{1j}, v_{2j}, v_{1j}\pm v_{2j}, v_{1j}\pm 2v_{2j}\}\},$$

for all $j \in \{k+1,\dots,n\}$. In addition, for $s_l \in \mathbb{N}_0$ with $0 \leq s_l \leq n$, where $l \in \{1,2,3\}$, consider the subset $\mathcal{V}^{n-k}_{(s_1,s_2,s_3)}$ of $\mathcal{V}^{n-k}$ as follows:

$$\begin{aligned}
\mathcal{V}^{n-k}_{(s_1,s_2,s_3)} : = \{\mathfrak{V}_{n-k} \in \mathcal{V}^{n-k}|\ &\mathrm{Card}\{V_j : V_j = v_{1j}\} = s_1, \mathrm{Card}\{V_j : V_j = v_{2j}\} = s_2,\\
&\mathrm{Card}\{V_j : V_j = v_{1j}\pm v_{2j}\} = s_3\}.
\end{aligned}$$

From now on, for a mapping $\Phi : \mathbb{V}^n \longrightarrow \mathbb{W}$, we remark

(A1) $\Phi\left(v_i^{[k]}, \mathcal{V}^{n-k}_{(s_1,s_2,s_3)}\right) := \sum_{\mathfrak{V}_{n-k}\in\mathcal{V}^{n-k}_{(s_1,s_2,s_3)}} \Phi\left(v_i^{[k]}, \mathfrak{V}_{n-k}\right)$;

(A2) $\Phi\left(\mathcal{V}^k_{(t_1,t_2)}, v_i^{[n-k]}\right) := \sum_{\mathfrak{V}_k\in\mathcal{V}^k_{(t_1,t_2)}} \Phi\left(\mathfrak{V}_k, v_i^{[n-k]}\right)$;

(A3) $\Phi\left(\mathcal{V}^k_{(t_1,t_2)},\mathcal{V}^{n-k}_{(s_1,s_2,s_3)}\right):=\sum_{\mathfrak{V}_k\in\mathcal{V}^k_{(t_1,t_2)}}\sum_{\mathfrak{V}_{n-k}\in\mathcal{V}^{n-k}_{(s_1,s_2,s_3)}}\Phi\left(\mathfrak{V}_k,\mathfrak{V}_{n-k}\right).$

In the upcoming result, we show that every multi-quintic-sextic mapping $\Phi:\mathbb{V}^n\longrightarrow\mathbb{W}$ can be represented as the equation

$$\begin{aligned}&2^k\sum_{q\in\{-1,1\}^n}\Phi\left(2v_1^{[n]}+qv_2^{[n]}\right)\\&=\sum_{t_1=0}^{k}\sum_{t_2=0}^{k-t_1}\sum_{s_1=0}^{n-k}\sum_{s_2=0}^{n-k-s_1}\sum_{s_3=0}^{n-k-s_1-s_2}(-1)^{n-t_1-t_2-s_1-s_2-s_3}\times 90^{t_1+s_1+s_2}\times 20^{t_2+s_3}\Phi\left(\mathcal{V}^k_{(t_1,t_2)},\mathcal{V}^{n-k}_{(s_1,s_2,s_3)}\right),\end{aligned}\tag{5}$$

for all $v_1^{[n]},v_2^{[n]}\in\mathbb{V}^n$, where $\Phi\left(\mathcal{V}^k_{(t_1,t_2)},\mathcal{V}^{n-k}_{(s_1,s_2,s_3)}\right)$ is defined in (A3).

Definition 2.2. We say a mapping $\Phi:\mathbb{V}^n\longrightarrow\mathbb{W}$ has the *m-power property* in the l^{th} variable if

$$\Phi(\omega_1,\dots,\omega_{l-1},2\omega_l,\omega_{l+1},\dots,\omega_n)=2^m\Phi(\omega_1,\dots,\omega_{l-1},\omega_l,\omega_{l+1},\dots,\omega_n),$$

for all $\omega_1,\dots,\omega_n\in\mathbb{E}$. 5-power and 6-power properties sometimes are called the *quintic* and the *sextic* properties.

For a mapping $\Phi:\mathbb{V}^n\longrightarrow\mathbb{W}$, we consider the following properties.

(P1) $\Phi\left(v^{[n]}\right)=0$ for any $v^{[n]}\in\mathbb{V}^n$ with at least one component which is equal to zero;

(P2) Φ has the quintic property in the first k variables;

(P3) Φ has the sextic property in the last $n-k$ components;

Remark 2.3. If a mapping $\Phi:\mathbb{V}^n\longrightarrow\mathbb{W}$ has properties (P2) and (P3), then property (P1) holds for it, but converse is not true in general.

In the next theorem, we give a representation of multi-quintic-sextic mappings.

Theorem 2.4. *Let $\Phi:\mathbb{V}^n\longrightarrow\mathbb{W}$ be a k-quintic and $n-k$-sextic mapping. Then, Γ fulfills equation* (5)*. The converse is true if Φ has properties* (P2) *and* (P3)*.*

Proof. For $k\in\{0,n\}$ the result follows from [20, Theorem 2.2] and [19, Theorem 2]. We now obtain the result when $k\in\{1,\dots,n-1\}$. For any $v^{[n-k]}\in\mathbb{V}^{n-k}$, define the mapping $\Phi_{v^{[n-k]}}:\mathbb{V}^k\longrightarrow\mathbb{W}$ by $\Phi_{v^{[n-k]}}\left(v^{[k]}\right)=\Phi\left(v^{[k]},v^{[n-k]}\right)$ for all $v^{[k]}\in\mathbb{V}^k$. By the assumption, $\Phi_{v^{[n-k]}}$ is k-quintic and so one can find from [19, Theorem 2] that

$$2^k\sum_{q\in\{-1,1\}^k}\Phi_{v^{[n-k]}}\left(2v_1^{[k]}+qv_2^{[k]}\right)=\sum_{t_1=0}^{k}\sum_{t_2=0}^{k-t_1}(-1)^{k-t_1-t_2}\times 90^{t_1}\times 20^{t_2}\Phi_{v^{[n-k]}}\left(\mathcal{V}^k_{(t_1,t_2)}\right),$$

for all $v_1^{[k]},v_2^{[k]}\in\mathbb{V}^k$. By the last equality, we have

$$2^k\sum_{q\in\{-1,1\}^k}\Phi\left(2v_1^{[k]}+qv_2^{[k]},v^{[n-k]}\right)=\sum_{t_1=0}^{k}\sum_{t_2=0}^{k-t_1}(-1)^{k-t_1-t_2}\times 90^{t_1}\times 20^{t_2}\Phi\left(\mathcal{V}^k_{(t_1,t_2)},v^{[n-k]}\right),\tag{6}$$

for all $v_1^{[k]}, v_2^{[k]} \in \mathbb{V}^k$ and $v^{[n-k]} \in \mathbb{V}^{n-k}$. Further, for any $v^{[k]} \in \mathbb{V}^k$, define the mapping $\Phi_{v^{[k]}} : \mathbb{V}^{n-k} \longrightarrow \mathbb{W}$ via $\Phi_{v^{[k]}}(v^{[n-k]}) := \Phi\left(v^{[k]}, v^{[n-k]}\right)$, where $v^{[n-k]} \in \mathbb{V}^{n-k}$. Our assumption implies that Φ is an $n-k$-sextic mapping and therefore by [20, Theorem 2.2], we get

$$\begin{aligned} &\sum_{q\in\{-1,1\}^{n-k}} \Phi_{v^{[k]}}\left(2v_1^{[n-k]} + qv_2^{[n-k]}\right) \\ &= \sum_{s_1=0}^{n} \sum_{s_2=0}^{n-k-s_1} \sum_{s_3=0}^{n-k-s_1-s_2} (-1)^{n-k-s_1-s_2-s_3} \times 90^{s_1+s_2} \times 20^{s_3} \Phi_{v^{[k]}}\left(\mathcal{V}^{n-k}_{(s_1,s_2,s_3)}\right), \end{aligned} \tag{7}$$

for all $v_1^{[n-k]}, v_2^{[n-k]} \in \mathbb{V}^{n-k}$. By the definition of $\Phi_{v^{[k]}}$ and equality (7), we obtain

$$\begin{aligned} &\sum_{q\in\{-1,1\}^{n-k}} \Phi\left(v^{[k]}, 2v_1^{[n-k]} + qv_2^{[n-k]}\right) \\ &= \sum_{s_1=0}^{n} \sum_{s_2=0}^{n-k-s_1} \sum_{s_3=0}^{n-k-s_1-s_2} (-1)^{n-k-s_1-s_2-s_3} \times 90^{s_1+s_2} \times 20^{s_3} \Phi\left(v^{[k]}, \mathcal{V}^{n-k}_{(s_1,s_2,s_3)}\right), \end{aligned} \tag{8}$$

for all $v_1^{[n-k]}, v_2^{[n-k]} \in \mathbb{V}^{n-k}$ and $v^{[k]} \in \mathbb{V}^k$. Plugging (6) and (8), we reach to (5).
Conversely, assume that Γ satisfies (5). It follows from Remark 2.3 that (P1) is valid for Φ. Putting $x_2^{[n-k]} = \mathbf{0}_{n-k}$ in the left side of (5) and using (P3), we get

$$2^k \times 2^{6(n-k)} \times 2^{n-k} \sum_{q\in\{-1,1\}^k} \Phi\left(2v_1^{[k]} + qv_2^{[k]}, v_1^{[n-k]}\right) = 2^{7n-6k} \sum_{q\in\{-1,1\}^k} \Phi\left(2v_1^{[k]} + qv_2^{[k]}, v_1^{[n-k]}\right), \tag{9}$$

where $\mathbf{0}_m = \left(\overbrace{0, \dots, 0}^{m-\text{times}}\right)$. The same replacement (as in the above) in the right side of (5) can be repeated to obtain

$$\begin{aligned} &\sum_{t_1=0}^{k} \sum_{t_2=0}^{k-t_1} \sum_{s_1=0}^{n-k} \sum_{s_3=0}^{n-k-s_1} \binom{n-k}{s_1} \binom{n-k-s_1}{s_3} \\ &(-1)^{n-t_1-t_2-s_1-s_3} \times 2^{n-s_1-s_3} \times 90^{t_1+s_1} \times 20^{t_2+s_3} \times 2^{s_3} \Phi\left(\mathcal{V}^k_{(t_1,t_2)}, v_1^{n-k}\right) \\ &= \sum_{t_1=0}^{k} \sum_{t_2=0}^{k-t_1} \sum_{s_1=0}^{n-k} \binom{n-k}{s_1} (-1)^{k-t_1-t_2} \times 38^{n-k-s_1} \times 90^{t_1+s_1} \times 20^{t_2} \Phi\left(\mathcal{V}^k_{(t_1,t_2)}, v_1^{n-k}\right) \\ &= 2^{7(n-k)} \sum_{t_1=0}^{k} \sum_{t_2=0}^{k-t_1} (-1)^{k-t_1-t_2} \times 90^{t_1} \times 20^{t_2} \Phi\left(\mathcal{V}^k_{(t_1,t_2)}, v_1^{n-k}\right). \end{aligned} \tag{10}$$

Comaring relations (9) and (10), we arrive at

$$2^k \sum_{q\in\{-1,1\}^k} \Phi\left(2v_1^{[k]} + qv_2^{[k]}, v_1^{[n-k]}\right) = \sum_{t_1=0}^{k} \sum_{t_2=0}^{k-t_1} (-1)^{k-t_1-t_2} \times 90^{t_1} \times 20^{t_2} \Phi\left(\mathcal{V}^k_{(t_1,t_2)}, v_1^{n-k}\right),$$

for all $v_1^{[k]}, v_2^{[k]} \in \mathbb{V}^n$ and $v_1^{[n-k]} \in \mathbb{V}^{n-k}$. In view of [19, Theorem 2], we see that Φ is quintic in each of the k first variables. Once more, by putting $v_2^{[k]} = \mathbf{0}_k$ in (5), we get

$$\begin{aligned} &2^k \times 2^{5k} \times 2^k \sum_{q\in\{-1,1\}^k} \Phi\left(v_1^{[k]}, v_1^{[n-k]} + qv_2^{[n-k]}\right) \\ &= \sum_{t_1=0}^{k}\sum_{t_2=0}^{k-t_1}\sum_{s_1=0}^{n-k}\sum_{s_2=0}^{n-k-s_1}\sum_{s_3=0}^{n-k-s_1-s_2} \binom{k}{t_1}\binom{k-t_1}{t_2} \\ &(-1)^{n-t_1-t_2-s_1-s_2-s_3} \times 2^{k-t_1-t_2} \times 90^{t_1+s_1+s_2} \times 20^{t_2+s_3} \times 2^{t_2}\Phi\left(v_1^k, \mathcal{V}^{n-k}_{(s_1,s_2,s_3)}\right). \end{aligned}$$

Similar to the above and by a routine computation, one can show that

$$\begin{aligned} &\sum_{q\in\{-1,1\}^k} \Phi\left(v_1^{[k]}, v_1^{[n-k]} + qv_2^{[n-k]}\right) \\ &= \sum_{s_1=0}^{n-k}\sum_{s_2=0}^{n-k-s_1}\sum_{s_3=0}^{n-k-s_1-s_2-s_3} (-1)^{n-k-s_1-s_2-s_3} \times 90^{s_1+s_2} \times 20^{s_3}\Phi\left(v_1^k, \mathcal{V}^{n-k}_{(s_1,s_2,s_3)}\right), \end{aligned}$$

for all $v_1^{[k]} \in \mathbb{V}^n$ and $v_1^{[n-k]}, v_2^{[n-k]} \in \mathbb{V}^{[n-k]}$. By the hypothesis and [20, Theorem 2.2], Φ is sextic in each of the last $n-k$ variables. The proof is now complete. ■

3 Stability results for (5)

In this section, we prove the Hyers stability of equation (5) by the direct method in the setting of Banach spaces.

For a mapping $\Phi : \mathbb{V}^n \longrightarrow \mathbb{W}$, we use the difference operation $\Gamma : \mathbb{V}^n \times \mathbb{V}^n \longrightarrow \mathbb{W}$ as follows:

$$\begin{aligned} \Gamma\Phi\left(v_1^{[n]}, v_2^{[n]}\right) &:= 2^k \sum_{q\in\{-1,1\}^n} \Phi\left(2v_1^{[n]} + qv_2^{[n]}\right) \\ &- \sum_{t_1=0}^{k}\sum_{t_2=0}^{k-t_1}\sum_{s_1=0}^{n-k}\sum_{s_2=0}^{n-k-s_1}\sum_{s_3=0}^{n-k-s_1-s_2} (-1)^{n-t_1-t_2-s_1-s_2-s_3} \times 90^{t_1+s_1+s_2} \times 20^{t_2+s_3}\Phi\left(\mathcal{V}^k_{(t_1,t_2)}, \mathcal{V}^{n-k}_{(s_1,s_2,s_3)}\right), \end{aligned}$$

for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n$, where $\Phi\left(\mathcal{V}^k_{(t_1,t_2)}, \mathcal{V}^{n-k}_{(s_1,s_2,s_3)}\right)$ is defined in (A3).

In the next theorem, we bring Hyers stability result for equation (5) by the direct way which is our main result in this section.

Theorem 3.1. *Let $\delta \in [0,\infty)$, $\mathbb{V}$ be a linear space over the rationals and $\mathbb{W}$ be a Banach space. Suppose that $\Phi : \mathbb{V}^n \longrightarrow \mathbb{W}$ is a mapping satisfies* (P1) *such that*

$$\left\|\Gamma\Phi\left(v_1^{[n]}, v_2^{[n]}\right)\right\| \leq \delta, \tag{11}$$

for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n$. Then, there exists a solution $\mathcal{F}_q^s : \mathbb{V}^n \longrightarrow \mathbb{W}$ of (5) *such that*

$$\left\|\Phi\left(v^{[n]}\right) - \mathcal{F}_q^s\left(v^{[n]}\right)\right\| \leq \frac{\delta}{2^{n+k}(2^{6n-k}-1)}, \tag{12}$$

for all $v^{[n]} \in \mathbb{V}^n$. *Moreover, if* $\mathcal{F}_q^s$ *fulfills* (P2) *and* (P3), *then it is a unique multi-quintic-sextic mapping satisfying* (12). *In particular,*

(i) if $k = n$, *then there exists a unique multi-quintic mapping* $\mathcal{Q} : \mathbb{V}^n \longrightarrow \mathbb{W}$ *such that*

$$\left\|\Phi\left(v^{[n]}\right) - \mathcal{Q}\left(v^{[n]}\right)\right\| \leq \frac{\delta}{2^{2n}(2^{5n}-1)},$$

for all $v^{[n]} \in \mathbb{V}^n$;

(ii) if $k = 0$, *then there exists a unique multi-sextic mapping* $\mathcal{S} : \mathbb{V}^n \longrightarrow \mathbb{W}$ *such that*

$$\left\|\Phi\left(v^{[n]}\right) - \mathcal{S}\left(v^{[n]}\right)\right\| \leq \frac{\delta}{2^{n}(2^{6n}-1)},$$

for all $v^{[n]} \in \mathbb{V}^n$.

Proof. Putting $v_2^{[n]} = \mathbf{0}_n$ in (11), we have

$$\left\|2^k \times 2^n \Phi\left(2v_1^{[n]}\right) - \lambda\mu\Phi\left(v_1^{[n]}\right)\right\| \leq \delta, \tag{13}$$

for all $v_1^{[n]} \in \mathbb{V}^n$, where

$$\lambda = \sum_{t_1=0}^{k}\sum_{t_2=0}^{k-t_1} \begin{pmatrix} k \\ t_1 \end{pmatrix} \begin{pmatrix} k-t_1 \\ t_2 \end{pmatrix} (-1)^{k-t_1-t_2} \times 2^{k-t_1-t_2} \times 90^{t_1} \times 20^{t_2} \times 2^{t_2}$$

and

$$\mu = \sum_{s_1=0}^{n-k}\sum_{s_3=0}^{n-k-s_1} \begin{pmatrix} n-k \\ s_1 \end{pmatrix} \begin{pmatrix} n-k-s_1 \\ s_3 \end{pmatrix} (-1)^{n-k-s_1-s_3} \times 2^{n-k-s_1-s_3} \times 90^{s_1} \times 20^{s_3} \times 2^{s_3}$$

By a routine computation, One can find that $\lambda = 2^{7k}$ and $\mu = 2^{7n-k}$. For the rest of proof, we use $v^{[n]}$ instead of $v_1^{[n]}$. It follows from relation (13) and last equalities that

$$\left\|2^{n+k}\Phi\left(2v^{[n]}\right) - 2^{7n}\Phi\left(v^{[n]}\right)\right\| \leq \delta,$$

for all $v^{[n]} \in \mathbb{V}^n$, and so the above inequality can be rewritten as

$$\left\|\frac{\Phi\left(2v^{[n]}\right)}{2^{6n-k}} - \Phi\left(v^{[n]}\right)\right\| \leq \frac{\delta}{2^{7n}}, \tag{14}$$

for all $v^{[n]} \in \mathbb{V}^n$. Substituting $v^{[n]}$ into $2v^{[n]}$ in (14) and repeating it, we obtain

$$\left\|\frac{\Phi\left(2^m v^{[n]}\right)}{2^{(6n-k)m}} - \frac{\Phi\left(2^l v^{[n]}\right)}{2^{(6n-k)l}}\right\| \leq \frac{\delta}{2^{7n}} \sum_{j=l}^{m-1} \frac{1}{2^{(6n-k)j}}, \tag{15}$$

for all $v^{[n]} \in \mathbb{V}^n$ and $m > l \geq 0$, and so the sequence $\left\{\frac{\Phi(2^m v^{[n]})}{2^{(6n-k)m}}\right\}$ is Cauchy by (15). Since the space $\mathbb{W}$ is Banach, there exists a mapping $\mathcal{F}_q^s : \mathbb{V}^n \longrightarrow \mathbb{W}$ such that

$$\lim_{m\to\infty} \frac{\Phi\left(2^m v^{[n]}\right)}{2^{(6n-k)m}} = \mathcal{F}_q^s\left(v^{[n]}\right), \qquad \left(v^{[n]} \in \mathbb{V}^n\right). \tag{16}$$

Letting $l = 0$ in (15), taking the limit as $m \to \infty$ and applying (16), we see that inequality (12) holds. Interchanging $\left(v_1^{[n]}, v_2^{[n]}\right)$ with $\left(2^m v_1^{[n]}, 2^m v_2^{[n]}\right)$ in (11) and dividing both sides by $2^{(6n-k)m}$, we obtain

$$\frac{1}{2^{(2n-k)m}} \left\|\Gamma\Phi\left(2^m v_1^{[n]}, 2^m v_2^{[n]}\right)\right\| \leq \frac{\delta}{2^{(2n-k)m}}.$$

Taking the limit as $m \to \infty$ in the last inequality and using (16), we arrive at $\Gamma\mathcal{F}_q^s\left(v_1^{[n]}, v_2^{[n]}\right) = 0$, for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{V}^n$, and thus $\mathcal{F}_q^s$ is a solution of (5). Assume now that $\mathcal{F}_q^s$ satisfies (P2) and (P3), then it is a multi-quintic-sextic mapping by Theorem 2.4. Let now $\mathfrak{F}_q^s : \mathbb{V}^n \longrightarrow \mathbb{W}$ be another multi-quintic-sextic mapping satisfying (12). Then, we have

$$\begin{aligned}
&\left\|\mathcal{F}_q^s\left(v^{[n]}\right) - \mathfrak{F}_q^s\left(v^{[n]}\right)\right\| \\
&= \frac{1}{2^{(6n-k)m}} \left\|\mathcal{F}_q^s\left(2^m v^{[n]}\right) - \mathfrak{F}_q^s\left(2^m v^{[n]}\right)\right\| \\
&\leq \frac{1}{2^{(6n-k)m}} \left(\left\|\mathcal{F}_q^s\left(2^m v^{[n]}\right) - \Phi\left(2^m v^{[n]}\right)\right\| + \left\|\Phi\left(2^m v^{[n]}\right) - \mathfrak{F}_q^s\left(2^m v^{[n]}\right)\right\|\right) \\
&\leq \frac{2}{2^{(6n-k)m}} \times \frac{\delta}{2^{n+k}(2^{6n-k} - 1)},
\end{aligned}$$

for all $v^{[n]} \in \mathbb{V}^n$. Letting $m \to \infty$ in the above inequality, we have $\mathcal{F}_q^s = \mathfrak{F}_q^s$ and therefore the solution is unique. This finishes the proof. ■

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