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Research Article

Thermal Analysis of a Porous Fin via Optimized Chebyshev polynomial with interior point algorithm

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Abstract

Abstract: An investigation has been conducted to examine the complexities associated with the thermal performance of a nonlinear problem pertaining to the porous fin characterized by temperature-dependent internal heat generation. It is posited that the heat generation is contingent upon temperature. The impacts of the natural convection parameter **Nc**, internal heat generation $\varepsilon\mathbf{g}$, porosity **Sh**, and generation number G on the dimensionless temperature distribution are thoroughly examined. A novel intelligent computational strategy is established for solution identification. To achieve this objective, the governing equation is reformulated into a corresponding problem with boundary conditions conducive to the application of a modified version of Chebyshev polynomials of the first kind. The functions based on these Chebyshev polynomials generate an approximate series solution with undetermined weights. The mathematical optimization framework comprises an unsupervised error, minimized by adjusting weights through the interior point method. The proposed approximate solution is corroborated by enforcing tolerance constraints within the optimization framework.
Keywords: Chebyshev polynomial of the first kind, Interior point method, porous fin, temperature-dependent heat generation.
2020 Mathematics Subject Classification: 34B15; 34B60

1 Introduction

Fins are commonly employed in numerous thermal transfer applications to enhance efficiency. Conversely, for an extended period, achieving elevated rates of heat transfer while minimizing the dimensions and expenditures of fins has been the primary objective for various engineering applications, including heat exchangers, economizers, conventional furnaces, and gas turbines, among others. Certain engineering applications, such as those in aviation and motorcycling, necessitate the utilization of lighter fins that facilitate an augmented rate of heat transfer. The augmentation of heat transfer is predominantly contingent upon the heat transfer coefficient (h), the available surface area, and the temperature gradient between the surface and the adjacent fluid. However,

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this stipulation is frequently rationalized by the considerable expense associated with high-thermal-conductivity metals, wherein metals exhibiting elevated thermal conductivity are also characterized by substantial costs. The fin is designed with a porous structure to facilitate the movement of infiltrating fluid through its matrix. A significant volume of scholarly research has been conducted in this domain, with numerous references accessible, particularly concerning heat transfer mechanisms in porous fins [1, 2]. Below, a selection of pertinent scholarly articles related to the investigation presented herein is delineated. Nonlinear issues and phenomena are of paramount significance in applied mathematics, physics, engineering, and other scientific disciplines, particularly in relation to certain heat transfer equations. With the exception of a limited subset of these issues, the majority lack definitive analytical solutions. Consequently, these nonlinear equations necessitate resolution through approximation methodologies. Perturbation techniques exhibit a pronounced dependence on the so-called small parameters [3]. Various alternative methodologies have been introduced to address nonlinear equations, including the -expansion method [4], Adomian's decomposition method [5], multiple homotopy perturbation methods (HPM) [6, 7, 8] numerous variational iteration methods (VIM) [9, 10], and various collocation methods [11, 12]. In the present study, we have employed the Chebyshev polynomials of the first kind to derive approximate solutions for the nonlinear differential equations governing the behavior of a porous fin with temperature-dependent internal heat generation. The findings reveal that the proposed methodology are both straightforward and precise in comparison to numerical approaches. It is observed that methodology serve as potent mathematical instrument and can be effectively applied across a broad spectrum of linear and nonlinear challenges encountered in various scientific and engineering fields. The computational expense and complexity inherent in the analysis of such problems are noteworthy. In this manuscript, we introduce a novel intelligent computational strategy aimed at deriving solutions for the nonlinear second-order boundary value problem articulated in Eq. (4). Initially, we reformulate the governing equation into an equivalent problem characterized by boundary conditions of $[-1, 1]$. This reformulation facilitates the convenient application of a reformed version of Chebyshev polynomials of the first kind. Subsequently, we optimize the Chebyshev polynomials of the first kind to generate an approximate series solution with undetermined weights. Moreover, an optimization problem is established based on an unsupervised error functioning as the objective function, subject to a tolerance constraint. This optimization problem is addressed through minimization by adjusting the weights via the interior point method.

2 Problem formulation

As shown in Figure 1, a rectangular porous fin profile is considered. The dimensions of this fin are length L, with wand thickness t. The cross-section area of the fin is constant and the fin has a temperature-dependent internal heat generation.

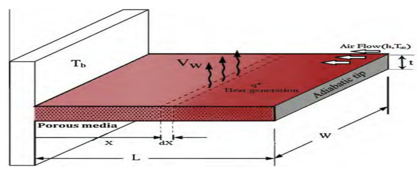

Fig. 1: Schematic of the longitudinal straight fin geometry with the internal heat generation.

Also, the heat loss from the tip of the fin compared with the top and bottom surfaces of the fin is assumed to be negligible. Since the transverse Biot number should be small for the fin to be effective [13], the temperature variation in the transverse direction is neglected. Thus heat conduction is assumed to occur solely in the longitudinal direction. The energy balance can be written as:

$$q(x) - q(x+\Delta x) + q^* A \Delta x = \dot{m} c_p [T(x) - T_\infty] + h(p.\Delta x)[T(x) - T_\infty] \tag{1}$$

The mass flow rate of the fluid passing through the porous material can be written as:

$$\dot{m} = \rho V_w \Delta x w \tag{2}$$

The value of V_w should be estimated from the consideration of the flow in the porous medium. From Darcy's model we have:

$$V_w = \frac{gk\beta}{\upsilon}[T(x) - T_\infty] \tag{3}$$

Substitution of Equations (2) and (3) into Equation (1) yields:

$$\frac{q(x) - q(x+\Delta x)}{\Delta x} + q^* A = \frac{\rho c_p g k \beta w}{\Delta x}[T(x) - T_\infty]^2 + hp[T(x) - T_\infty] \tag{4}$$

As $\Delta x \to 0$, Exquation (4) becomes

$$\frac{dq}{dx} + q^* A = \frac{\rho c_p g k \beta w}{\Delta x}[T(x) - T_\infty]^2 + hp[T(x) - T_\infty] \tag{5}$$

Also from Fourier's law of conduction:

$$q = -k_{eff} A \frac{dT}{dx} \tag{6}$$

where A is the cross-sectional area of the fin $A = wt$ and k_{eff} is the effective thermal conductivity of the porous fin that can be obtained from the following equation:

$$k_{eff} = \varphi k_f + (1-\varphi) k_s \tag{7}$$

where φ is the porosity of the porous fin. Substitution of Equation (6) into Equation (5) leads to:

$$\frac{d^2T}{dx^2} - \frac{\rho c_p g k \beta w}{t k_{eff} \upsilon}[T(x) - T_\infty]^2 + \frac{hp}{k_{eff}A}[T - T_\infty] + \frac{q^*}{k_{eff}} = 0 \tag{8}$$

It is assumed that heat generation in the fin varies with temperature as Equation (9):

$$q^* = q^*_\infty[1 + \varepsilon(T - T_\infty)] \tag{9}$$

where q^*_∞ is the internal heat generation at temperature T_∞. For simplifying the above equations, some dimensionless parameters are introduced as follows:

$$x = \frac{X}{L}, \qquad \theta = \frac{T - T_\infty}{T_b - T_\infty}, \qquad Nc^2 = \frac{hpL^2}{k_0 A},$$
$$Sh = \frac{DaxRa}{kr}\left(\frac{L}{t}\right)^2, \qquad G = \frac{q^*_\infty}{hp(T_b - T_\infty)}$$
$$\varepsilon g = \varepsilon(T_b - T_\infty). \tag{10}$$
$$\tag{11}$$

where Sh is a porous parameter that indicates the effect of the permeability of the porous medium as well as the buoyancy effect, so a higher value of Sh indicates higher permeability of the porous medium or higher buoyancy forces. Nc is a convection parameter that indicates the effect of surface convecting of the fin. Finally, Equation (8) can be rewritten as:

$$\frac{d^2\theta}{dx^2} - Nc^2\theta + Nc^2G(1 + \varepsilon g\theta) - Sh\theta^2 = 0 \tag{12}$$

In this research, we study a finite-length fin with an insulated tip. For this case, the fin tip is insulated so that there will not be any heat transfer at the insulated tip and boundary condition will be

$$x = 1 : \theta = 1, \qquad x = 0 : \frac{d\theta}{dx} = 0. \tag{13}$$

3 High order derivatives of Basis functions

Chebyshev polynomials [14] are very useful as orthogonal polynomials on the interval $[-1, 1]$ of the real line. These polynomials have very good properties in the approximation of functions so that appear frequently in several fields of mathematics, physics and engineering.

3.1 Basic Properties of Chebyshev Polynomials

The Chebyshev polynomials of the first kind, known as $T_n(x) = \cos(n \arccos x)$, can be obtained by means of Rodrigue's formula [15]

$$T_n(x) = \frac{\Gamma(\frac{1}{2})}{(-2)^n\Gamma(n + \frac{1}{2})}\sqrt{1 - x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(1 - x^2\right)^{n-\frac{1}{2}}, \quad n = 0, 1, 2, \cdots. \tag{14}$$

The Chebyshev polynomials of the first kind can be developed by means of the generating function too, as follows:

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{+\infty} T_n(x)t^n. \tag{15}$$

The first two Chebyshev polynomials $T_n(x) = 1$ and $T_n(x) = x$ are known from (14), all other polynomials $T_n(x), n \geq 2$ can be obtained by means of the recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{16}$$

The derivative of $T_n(x)$ with respect to x can be obtained from

$$\left(1-x^2\right) T_n'(x) = -nxT_n(x) + nT_{n-1}(x),\ x \neq \pm 1, \tag{17}$$

$$T_n'(-1) = n^2(-1)^{n+1}, \quad T_n'(1) = n^2. \tag{18}$$

The following special values and properties of $T_n(x)$ are well established and will be useful:

$$T_n(-x) = (-1)^n T_n(x),\ T_n(1) = 1,\ T_n(-1) = (-1)^n,\ T_{2n}(0) = (-1)^n,\ T_{2n+1}(0) = 0. \tag{19}$$

We can determine the orthogonality properties for the Chebyshev polynomials of the first kind from our knowledge of the orthogonality of the cosine functions, as

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} \mathrm{d}x = \begin{cases} 0, & m \neq n; \\ \frac{\pi}{2}, & m = n \neq 0; \\ \pi, & m = n = 0. \end{cases} \tag{20}$$

We observe that the Chebyshev polynomials form an orthogonal set on the interval $[-1, 1]$ with the weighting function $\frac{1}{\sqrt{1-x^2}}$.

3.2 High Order Derivatives of Chebyshev Polynomials

(The Leibniz Formula) For a function $f(x) = g(x)h(x)$, the derivatives of $f(x)$ can be represented as a sum of derivatives of $g(x)$ and $h(x)$ as:

$$f^{(k)}(x) = \sum_{n=0}^{k} \binom{k}{n} g^{(n)}(x) h^{(k-n)}(x), \tag{21}$$

where $\binom{k}{n}$ are the binomial coefficients.

Theorem 3.1. ***(Slevinsky-Safouhi)[16]*** *Let $G(x)$ be a function kth differentiable and with the term $\left(\frac{d}{x\,dx}\right)^k G(x)$ welldefined. The term $\frac{d^kG}{dx^k}$ is given by:*

$$\frac{d^kG}{dx^k} = \sum_{i=\lfloor\frac{k+1}{2}\rfloor}^{k} \hat{A}_k^i x^{2i-k} \left(\frac{d}{x\,dx}\right)^k G(x), \tag{22}$$

with coefficients:

$$\hat{A}_k^i = \begin{cases} 1, & i{=}k; \\ 2\hat{A}_{k-1}^i + \hat{A}_{k-1}^{i-1}, & i = \lfloor\frac{k+1}{2}\rfloor, k \text{ odd}; \\ \hat{A}_{k-1}^i, & i = \lfloor\frac{k+1}{2}\rfloor, k \text{ even}; \\ (2i-k+1)\hat{A}_{k-1}^i + \hat{A}_{k-1}^{i-1}, & \lfloor\frac{k+1}{2}\rfloor < i < k, k > 3. \end{cases} \tag{23}$$

where $\lfloor\alpha\rfloor$ is the integer floor function of argument α.

It is natural with the help of the Leibniz Formula as well as Rodrigue's formula to define the higher order derivatives of $T_n(x)$ as:

$$\frac{\mathrm{d}^i}{\mathrm{d}x^i}T_n(x) = \frac{\Gamma(\frac{1}{2})}{(-2)^n\Gamma(n+\frac{1}{2})}\sum_{l=0}^{i}\binom{i}{l}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\sqrt{1-x^2}\,\frac{\mathrm{d}^{n+i-l}}{\mathrm{d}x^{n+i-l}}\left(1-x^2\right)^{n-\frac{1}{2}}. \tag{24}$$

Without going into great detail, if we apply the result of Theorem 3.1 into the above equation then we develop a very effective formula as the final result [16]:

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}x^k}T_n(x) \quad = \quad & \frac{\Gamma(\frac{1}{2})}{(-2)^n\Gamma(n+\frac{1}{2})}\sum_{l=0}^{k}\Bigg\{\binom{k}{l}\left[\sum_{i=\lfloor\frac{l+1}{2}\rfloor}^{l}\hat{A}_l^i x^{2i-l}(-2)^i\left(1-x^2\right)^{\frac{1}{2}-i}\prod_{j=0}^{i-1}\left(\frac{1}{2}-j\right)\right]\times \\ & \left[\sum_{i=\lfloor\frac{n+k-l+1}{2}\rfloor}^{n+k-l}\hat{A}_{n+k-l}^i x^{2i-n-k+l}(-2)^i\left(1-x^2\right)^{n-\frac{1}{2}-i}\prod_{j=0}^{i-1}\left(n-\frac{1}{2}-j\right)\right]\Bigg\}\end{aligned} \tag{25}$$

with coefficients $\hat{A}_k^i$ given by (23).

4 Proposed Method

By the change of variable $x \mapsto \frac{1}{2}x + \frac{1}{2}$, the boundary value problem Eqs. (10)-(13) can be rewritten as

$$4\frac{d^2\theta}{dx^2} - Nc^2\theta + Nc^2G(1+\varepsilon g\theta) - Sh\theta^2 = 0, \tag{26}$$

$$\theta(1) = 1, \theta'(-1) = 0. \tag{27}$$

Now, it is convenient to treat them by Chebyshev polynomials of the first kind. Moreover, the change of function $\theta \mapsto \theta + 1$ transforms the problems into

$$4\frac{d^2\theta}{dx^2} - Nc^2(\theta+1) + Nc^2G(1+\varepsilon g(\theta+1)) - Sh(\theta+1)^2 = 0, \tag{28}$$

$$\theta(1) = 0, \theta'(-1) = 0, \tag{29}$$

such that the boundary conditions become homogenous.

4.1 Reformed Version of Chebyshev Polynomials

Define $\hat{T}_n,\ n \geq 1$ as

$$\hat{T}_n(x) = T_n(x) - (-1)^{n+1}n^2x + (-1)^{n+1}n^2 - 1, \quad n \geq 1, \tag{30}$$

then obviously, from (19), we have

$$\hat{T}_n(1) = 0, \quad n \geq 1. \tag{31}$$

Eq. (18) implies

$$\hat{T}'_n(-1) = T'_n(-1) - (-1)^{n+1}n^2 = 0 \quad n \geq 1. \tag{32}$$

Therefore, from Eqs. (31)-(32), we conclude that the boundary conditions (29) hold.

Furthermore, the second derivative of the reformed version of Chebyshev polynomials of the first kind are given by

$$\hat{T}'_n(x) = T'_n(x) - (-1)^{n+1}n^2, \quad n \geq 1, \tag{33}$$

$$\hat{T}''_n(x) = T''_n(x), \quad n \geq 1, \tag{34}$$

where the right hand side can be obtained by the formula (25) when $k = 1, 2$.

4.2 Corresponding Optimization Problem

Define a approximate series solution of order M as

$$\Theta_M(x) = \sum_{n=1}^{M} \alpha_n \hat{T}_n(x), \tag{35}$$

and consider the number of N regularly distributed nodal points in interval $[-1, 1]$, namely $x_i, i = 1, 2, ..., N$, then we define the unsupervised errors as the sum of mean squared errors:

$$\begin{aligned} &\epsilon(N, \alpha) = \\ &\frac{1}{N}\sum_{i=1}^{N}\Bigg\{4\sum_{n=1}^{M}\alpha_n\hat{T}''_n(x_i) - Nc^2(\sum_{n=1}^{M}\alpha_n\hat{T}_n(x_i)+1) + Nc^2G(1+\varepsilon g(\sum_{n=1}^{M}\alpha_n\hat{T}_n(x_i)+1)) \\ &-Sh(\sum_{n=1}^{M}\alpha_n\hat{T}_n(x_i)+1)^2\Bigg\}^2, \end{aligned} \tag{36}$$

It is worth to mention here that $\Theta_M(x)$ automatically satisfy boundary conditions (29). Now, define the following optimization problems

$$\begin{aligned} &\min_{\alpha} \epsilon(N, \alpha) \\ &\text{subject to} \quad \epsilon(N, \alpha) - \varepsilon \leq 0, \end{aligned} \tag{37}$$

where ε is a given tolerance. In our approach, the interior point method (IPM) is used for tuning of weights of the approximate series solution (35). IPM belongs to a class of algorithms which are used for treating constrained optimization problems. The technique is based on Karmarkar's algorithm which has been developed by Narendra Karmarkar in 1984 for linear programming resolution [17]. Detailed in formation about the algorithm is available in references [18, 19]. IPMs have been applied to many optimization problems in engineering and applied science such as multi-area optimal reactive power flow [20] and economic dispatch problem [21]. The fundamental trait of interior point methods are based on self-concordant barrier functions which play important role in encoding the convex set. In contrast to the classical simplex method, search for an optimal solution is made by traversing the interior of the feasible region and solving a sequence of subproblems [22].

5 Numerical experiments and comparison

In this section, we show the results obtained for some case studies which have been adopted from Refs. [23, 24, 25, 26] using proposed method described in the previous sections. In these examples, $N = 50$, the number of total nodal points covering $[-1, 1]$, is regularly distributed. Moreover, the number of basis function in approximate series solution in Eq. (35) is $M = 14$. The obtained solutions can be compared to those of Refs. [23, 24, 25, 26] and references therein. All approximate solutions reported here obtained in seconds by MATLAB softwares programm, therefore the method is highly robust. MATLAB provides an efficient optimization toolbox that contains functions for finding minimum of a multi-variable function while satisfying constraints. The toolbox includes solvers that perform optimization on the various types of linear or nonlinear problems. The function, $fmincon(\cdot)$, of this toolbox is a general, multipurpose optimizer that well tested and frequently used to solve nonlinear programming problems with general equality, inequality, and bound constraints of small, medium, and large scale. To handle optimization problem Eq. (37), we use $fmincon(\cdot)$ augmented to the interior point method (IPM) as described in the previous section. Eq. (10) with the initial and boundary conditions Eqs. (12) and (13) was solved numerically using interior point method. The results are proven to be precise and accurate in solving a wide range of mathematical and engineering problems, especially fluid mechanic cases. This accuracy gives us high confidence about the validity of this problem and reveals an excellent agreement of engineering accuracy. This investigation is completed by depicting the effects of some important parameters the convection parameter, generation number, internal heat generation parameter and porous parameter evaluate how these parameters influence this temperature. From a physical point of view, Figures 2 to 5 are prepared in order to see the effects of the K and R flow parameters on the temperature distribution. As can be seen, the effect of natural convective heat loss (Nc) on non-dimensional temperature is shown in Figure 2. However, these figures show that as the buoyancy effects become stronger, i.e., Nc increases, the local temperature in the fin decreases.

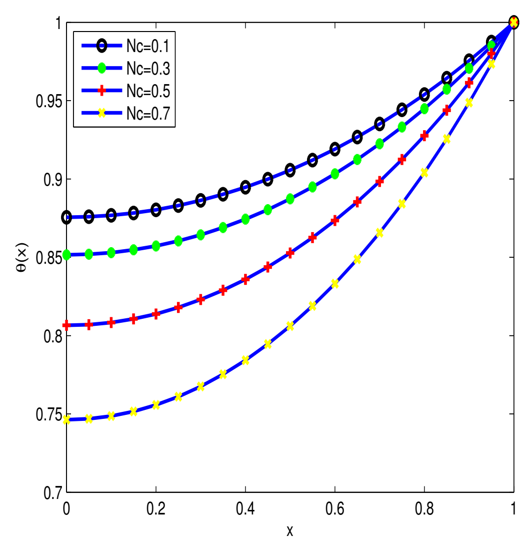

Fig. 2: Effect convective parameter and comparison of CM, HPM results with the numerical solution $Sh = 0.3, \varepsilon g = 0.4, G = 0.1$.

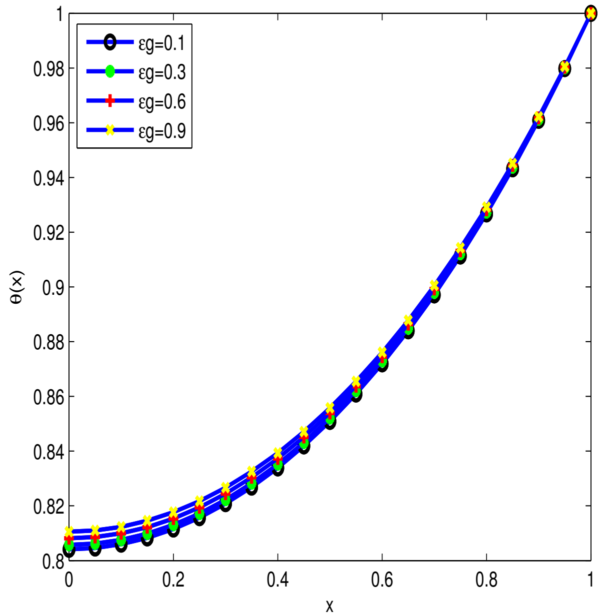

Fig. 3: Effect of internal heat generation parameter and comparison of CM, HPM results with the numerical solution $Sh = 0.3, Nc = 0.5, G = 0.1$.

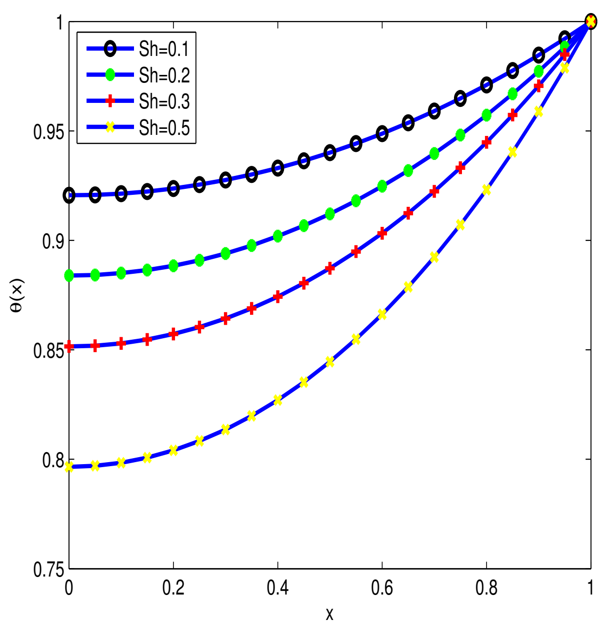

Fig. 4: Effect of porous parameter and comparison of CM, HPM results with the numerical solution, β when $\varepsilon g = 0.4, Nc = 0.3, G = 0.1$.

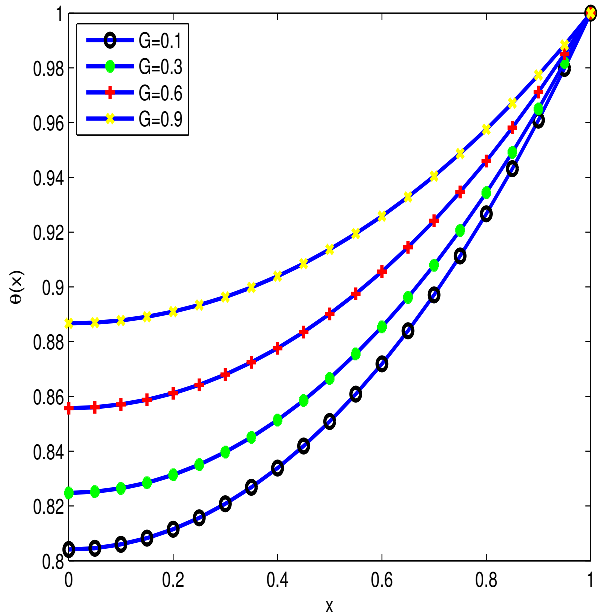

Fig. 5: Effect of generation number and comparison of CM, HPM results with the numerical solution, $\varepsilon g = 0.1, Nc = 0.5, Sh = 0.3$.

6 Conclusion

In this paper, For assessment of the CM and HPM, the solution and numerical tool, Table 1 has been presented. Comparison of the analytical solution with the numerical outcomes shows that the proposed methods are a convenient and powerful method in the engineering problem. It was also found that increasing Sh while increasing either Da or Ra increases the heat transfer from fin. In addition, as the buoyancy effects become stronger, i.e., Nc increases, the local temperature in the fin decreases. It has been proposed a new intelligent computational technique to obtain approximate solution for the mentioned problem. First, the governing equation is transformed into an equivalent problem whose boundary conditions are homogeneous in interval $[-1, 1]$. Then, it is optimized Chebyshev polynomials of the first kind to construct approximate series solution with unknown weights. Furthermore, by defining an optimization problem and minimizing it all weights are obtained via interior point method. It has been revealed through test studies that the method is highly robust and reliable.

Table 1: The results of HPM, CM, IPM and numerical methods for $\theta(x)$ for $Nc = 0.3$, $\varepsilon g = 0.2$, $Sh = 0.1$ and $G = 0.4$.

x	CM	HPM	NUM	IPM
0.00	0.934222193	0.934213444	0.934213428	0.934211514
0.05	0.934383116	0.934374229	0.934374223	0.934372310
0.10	0.934866626	0.934856727	0.934856715	0.934854854
0.15	0.935673835	0.935661244	0.935661229	0.935659456
0.20	0.936805856	0.936788323	0.936788309	0.936786616
0.25	0.938263800	0.93823873	0.938238716	0.938237076
0.30	0.940048779	0.940013444	0.940013429	0.940011823
0.35	0.942161907	0.942113664	0.942113650	0.942112077
0.40	0.944604294	0.944540815	0.944540802	0.944539278
0.45	0.947377052	0.947296545	0.947296532	0.947295076
0.50	0.950481295	0.950382725	0.950382714	0.950381337
0.55	0.953918134	0.953801462	0.953801451	0.953800152
0.60	0.957688681	0.957555090	0.957555079	0.957553853
0.65	0.961794048	0.961646179	0.961646169	0.961645022
0.70	0.966235347	0.966077540	0.966077531	0.966076493
0.75	0.971013691	0.970852227	0.970852218	0.970851346
0.80	0.976130192	0.975973539	0.975973531	0.975972889
0.85	0.981585961	0.981445028	0.981445023	0.971444646
0.90	0.987382110	0.987270513	0.987270505	0.987270352
0.95	0.993519753	0.993454067	0.993454050	0.993454007
1.00	1.000000000	1.000000000	1.000000000	1.000000000

	Nomenclature		
A	Section area of fin	εg	Internal heat generation
x	Dimensional space coordinates	Nc	Natural convection parameter
X	Horizontal direction	G	Generation number
h	Convection heat transfer coefficient	T_b	Fin base temperature
Kr	Thermal conductivity ratio	T	Local fin temperature
k	Thermal conductivity	T_∞	ambient temperature
q	Conducted heat	Sh	Porosity parameter
p	Fin perimeter	V_w	Velocity of fluid passing through the fin
CM	Collocation method		**Greeks**
HPM	Homotopy perturbation method	β	Coefficient of volumetric thermal expansion
NUM	Numerical method	ε	Fin surface emissivity (dimensionless)
IPM	Interior point method	θ	Dimensionless temperature
L	Length of the fin		**Subscripts**
c_p	Specific heat	eff	Porous properties
q^*	Heat generation	b	Conditions at the fin base

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