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Research Article

Solving linear optimization problems subject to bipolar fuzzy relational equalities defined with max-Hamacher family of t-norms

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Abstract

Abstract: This paper considers linear objective function optimization under the bipolar system of fuzzy relation equations constraints defined by the max-Hamacher family of t-norms, which is a parametric family of continuous strict t-norms whose members are decreasing functions of the parameters. It is demonstrated that the feasible solution set is represented as the union of a finite number of closed convex cells that are not necessarily connected. In order to determine the feasibility of the proposed system, some necessary and sufficient conditions are derived based on the bipolar FRE constraint defined by the max-Hamacher t-norm. Therefore, the feasible solution set for the problem is completely identified. Also, some simplification techniques have been introduced to accelerate the solution of the current problem, and an algorithm has been developed accordingly in order to identify feasible regions. To further clarify the approach presented in the paper, a step-by-step example is presented in several sections.
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1 Introduction

Fuzzy relational equation (FRE) is an expanded version of a Boolean relation equation, initially developed by Sanchez for medical diagnosis [1]. Since then, FRE has been applied to a wide range of

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fields. Accordingly, when the rules of inference are applied and their corresponding consequences are known, the problem of determining antecedents is simplified and mathematically reduced to solving an FRE [2]. Solvability identification and finding a set of solutions are primary and fundamental matters concerning FRE problems. Di Nola et al. proved that the solution set of FRE (if it is non-empty), defined by a continuous max-t-norm composition, is often a non-convex set. This non-convex set is completely determined by one maximum solution and a finite number of minimal solutions [3]. Non-convexity is one of two bottlenecks contributing to the complexity of FRE-related problems, particularly optimization problems involving fuzzy relations. In addition, finding minimal solutions to FREs is another bottleneck. Chen and Wang [4] presented an algorithm for obtaining the logical representation of all minimal solutions and indicated that a polynomial-time algorithm to find all minimal solutions to FRE may not exist. Also, Markovskii showed that solving max-product FRE is closely related to the covering problem, which is a type of NP-hard problem [5]. In fact, the same result holds for more general t-norms instead of the minimum and product operators [6], [7], [8].

Max-Hamacher t-norms are decreasing functions of the parameter α and each member of this family is in fact a strict t-norm [9]. In [10], some novel operational rules for hesitant fuzzy sets were presented by applying the Hamacher t-norm and t-conorm, in which a family of hesitant fuzzy Hamacher operators was proposed for aggregating hesitant fuzzy information. It was found that monotonic alternative scores can be obtained through the use of Hamacher arithmetic and geometric aggregation in [11]. As well, they investigated the relationship between alternative scores generated by Hamacher's arithmetic and geometric aggregate operators. As described in [12], the authors examined the general parametric Hamacher t-norm, in which the free parameter influences the quality of modeling and the learning capability of the system.

Bipolar fuzzy relation equations were also generalized by the FRE concept. The concept of bipolarity is deeply rooted in the human understanding of information and preferences [13]. Dubois and Prade have presented an overview of the asymmetric bipolar representation of positive and negative information in possibility theory [14]. They have shown that the possibility theory framework is convenient for handling bipolar representations, which were applied to distinguish between negative and positive information in preference modeling [13], [14]. The linear optimization of bipolar FRE was further the focus of a study conducted by some researchers where FRE was defined with max-min [15], [16], max-product [17], and max-Lukasiewicz composition [18], [19], [20], [21]. In [15], the concept of bipolar FRE was firstly proposed with a max-min composition where the constraints are expressed as $max_{j=1}^{n}\left\{max\left\{min\left\{a_{ij}^{+},x_j\right\},min\left\{a_{ij}^{-},1-x_j\right\}\right\}\right\} \quad$ for $i=1,\dots,m$, where $a_{ij}^{+},a_{ij}^{-},x_j\in[0,1]$. In a separate study, the described bipolar linear optimization problem was solved by an analytical method based on the resolution and some structural properties of the feasible region (using a necessary condition for characterizing an optimal solution and a simplification process for reducing the problem) [19]. However, a resolution method for obtaining the complete solution set of bipolar max-Lukasiewicz FRE was not found in the mentioned works [20]. In addition, in [22], the authors studied bipolar FREs defined by max-strict t-norms. Based on the constraints of the problem, they categorize them into four groups, each requiring a different approach to find the corresponding feasible solution set. Further, the article [23] considered a broad range of Archimedean t-norms. The basis of their analysis to resolve the problem was the use of the additive generator f_φ and pseudoinverse $f_\varphi^{(-1)}$ of each t-norm, and by utilizing the branch-and-bound method, they tried to prune non-optimal branches.

This paper studies bipolar FRE linear optimization problems using the following mathematical model:

$$\begin{array}{ll} min & c^T x \\ & A^+ \varphi x \vee A^- \varphi (\mathbf{1} - x) = b \\ & x \in [0,1]^n \end{array} \tag{1}$$

where $A^+ = \left(a_{ij}^+\right)_{m\times n}$ and $A^- = \left(a_{ij}^-\right)_{m\times n}$ are fuzzy matrices and $b = (b_i)_{m\times 1}$ is a fuzzy vector such that $0 \le a_{ij}^+ \le 1, 0 \le a_{ij}^- \le 1$ and $0 \le b_i \le 1$ for each $i \in \mathscr{I} = \{1, 2, \ldots, m\}$ and each $j \in \mathscr{J} = \{1, 2, \ldots, n\}$, respectively. Also, the constant vector $c = (c_j)_{n\times 1}$, the sum vector $\mathbf{1}$ (each component of the vector is equal to one) and the unknown vector $x = (x_j)_{n\times 1}$ are in $\mathbb{R}^n$. Moreover φ denotes max-Hamacher composition, that is,

$$\varphi(x, y) = T_H^\alpha(x, y) = \begin{cases} 0 & , \quad \alpha = x = y = 0 \\ \frac{xy}{\alpha + (1-\alpha)(x+y-xy)} & , \quad otherwise \end{cases}$$

in which $\alpha \ge 0$. The constraints in Problem (1) mean:

$$max_{j=1}^n \left\{ max \left\{ T_H^\alpha \left(a_{ij}^+, x_j\right), T_H^\alpha \left(a_{ij}^-, 1 - x_j\right) \right\} \right\} = b_i \quad , \forall i \in \mathscr{I} \tag{2}$$

and we have:

$$T_H^\alpha(a_{ij}^+, x_j) = \begin{cases} 0 & , \quad \alpha = a_{ij}^+ = x_j = 0 \\ \frac{a_{ij}^+ x_j}{\alpha + (1-\alpha)(a_{ij}^+ + x_j - a_{ij}^+ x_j)} & , \quad otherwise \end{cases}$$

and

$$T_H^\alpha(a_{ij}^-, 1 - x_j) = \begin{cases} 0 & , \quad \alpha = a_{ij}^- = (1 - x_j) = 0 \\ \frac{a_{ij}^- (1-x_j)}{\alpha + (1-\alpha)(a_{ij}^- + (1-x_j) - a_{ij}^- (1-x_j))} & , \quad otherwise \end{cases}$$

The following summarizes the rest of the paper. Section 2, presents a preliminary description of strict bipolar FREs using the Hamacher t-norm, including definitions, concepts, and properties. Section 3, presents the identification of the feasible solution set of the original problem, along with two derived necessary conditions for its feasibility. Further, a necessary and sufficient condition is provided to ensure that a given point is feasible for Problem (1) or not. Section 4 describes five simplification rules introduced to accelerate the resolution process by reducing the size of the problem, followed by an algorithm for the resolution of the feasible regions of the current optimization problems. In Section 5, the Problem (1) is solved by finding the local optima and the global optimal solution as the local optimum with the smallest objective function value, and in Section 6, the whole use of the approach is presented in a step-by-step example.

2 Preliminary definitions and properties

At first, this section will describe the feasible solution set for max $\left\{ T_H^\alpha \left(a_{ij}^+, x_j\right), T_H^\alpha \left(a_{ij}^-, 1 - x_j\right) \right\} = b_i$ where a^+, a^- and b are fixed scalars in $[0, 1]$, $x \in [0, 1]$ and T_H^α denote Hamacher t-norm. Then, the feasible solution set of the equation is completely characterized for each $i \in \mathscr{I}$ and each

$j \in \mathscr{J}$. Then, based on the results of this section, the feasible region of Problem (1) is determined in the next section. In order to ensure simplicity, let S_i be the feasible solution set of the i'th equation; that is, $S_i = \left\{x \in [0,1]^n : \max_{j=1}^{n} \left\{\max\left\{T_H^\alpha\left(a_{ij}^+, x_j\right), T_H^\alpha\left(a_{ij}^-, 1-x_j\right)\right\}\right\} = b_i\right\}$. As well, let $S(A^+, A^-, b)$ be the feasible solution set for Problem (1). Hence, it is clear that $S(A^+, A^-, b) = \bigcap_{i \in \mathscr{J}} S_i$.

Definition 2.1. For each $i \in \mathscr{J}$ and each $j \in \mathscr{J}$, we define $S_{ij}^+ = \{x \in [0,1] : T_H^\alpha(a_{ij}^+, x) = b_i\}$ and $S_{ij}^- = \{x \in [0,1] : T_H^\alpha(a_{ij}^-, 1-x) = b_i\}$; that is, S_{ij}^+ and S_{ij}^- denote the feasible solution sets of the equations $T_H^\alpha(a_{ij}^+, x) = b_i$ and $T_H^\alpha(a_{ij}^-, 1-x) = b_i$, respectively. Furthermore, define $I_{ij}^+ = \{x \in [0,1] : T_H^\alpha(a_{ij}^+, x) \le b_i\}$ and $I_{ij}^- = \{x \in [0,1] : T_H^\alpha(a_{ij}^-, 1-x) \le b_i\}$.

According to Definition 2.1, it is easy to verify that $S_{ij}^+ \subseteq I_{ij}^+$ and $S_{ij}^- \subseteq I_{ij}^-, \forall i \in \mathscr{I}$ and $\forall j \in \mathscr{J}$. Also, the following results are directly obtained from [24] and Definition 2.1:

$$S_{ij}^+ = \begin{cases} \{v_{ij}^+\} & , a_{ij}^+ \ge b_i \\ \varnothing & , a_{ij}^+ < b_i \end{cases} \qquad , I_{ij}^+ = \begin{cases} \left[0, v_{ij}^+\right] & , a_{ij}^+ \ge b_i \\ [0,1] & , a_{ij}^+ < b_i \end{cases}$$
$$S_{ij}^- = \begin{cases} \{1 - v_{ij}^-\} & , a_{ij}^- \ge b_i \\ \varnothing & , a_{ij}^- < b_i \end{cases} \qquad , I_{ij}^- = \begin{cases} \left[1 - v_{ij}^-, 1\right] & , a_{ij}^- \ge b_i \\ [0,1] & , a_{ij}^- < b_i \end{cases} \tag{3}$$

where $v_{ij}^+ = ([\alpha + (1-\alpha)a_{ij}^+]b_i)/(a_{ij}^+ - (1-\alpha)(1-a_{ij}^+)b_i)$ and $v_{ij}^- = ([\alpha + (1-\alpha)a_{ij}^-]b_i)/(a_{ij}^- - (1-\alpha)(1-a_{ij}^-)b_i)$.

Definition 2.2. For each $i \in \mathscr{I}$ and each $j \in \mathscr{J}$, we define

$$S_{ij} = \left\{x \in [0,1] : \max\left\{T_H^\alpha\left(a_{ij}^+, x\right), T_H^\alpha\left(a_{ij}^-, 1-x\right)\right\} = b_i\right\}$$

and

$$I_{ij} = \left\{x \in [0,1] : \max\left\{T_H^\alpha\left(a_{ij}^+, x\right), T_H^\alpha\left(a_{ij}^-, 1-x\right)\right\} \le b_i\right\}.$$

Lemma 2.3. ***(a)*** $I_{ij} = I_{ij}^+ \cap I_{ij}^-$. ***(b)*** $S_{ij} = I_{ij} \cap \left(S_{ij}^+ \cup S_{ij}^-\right)$.

Proof. **(a)** The proof is easily resulted from Definitions 2.1 and 2.2. **(b)** According to Definition 2.2, $x \in S_{ij}$ iff $T_H^\alpha\left(a_{ij}^+, x_j\right) \le b_i$, $T_H^\alpha\left(a_{ij}^-, 1-x_j\right) \le b_i$ (i.e., $x \in I_{ij}^+ \cap I_{ij}^- = I_{ij}$) and at least one of the two equalities $T_H^\alpha\left(a_{ij}^+, x_j\right) = b_i$ (i.e., $x \in S_{ij}^+$) and $T_H^\alpha\left(a_{ij}^-, 1-x_j\right)\Big\} = b_i$ (i.e., $x \in S_{ij}^-$) holds. ■

The following are the results obtained from (3) and Lemma 2.3, identifying S_{ij} and I_{ij} for all cases.

Corollary 2.4. *Suppose that $i \in \mathscr{I}, j \in \mathscr{J}$.*

- ***(a)*** *If $a_{ij}^+ < b_i$ and $a_{ij}^- < b_i$, then $I_{ij} = [0,1]$ and $S_{ij} = \varnothing$.*
- ***(b)*** *If $a_{ij}^+ \ge b_i$ and $a_{ij}^- < b_i$, then $I_{ij} = \left[0, v_{ij}^+\right]$ and $S_{ij} = \{v_{ij}^+\}$.*

(c) *If* $a_{ij}^- \geq b_i$ *and* $a_{ij}^+ < b_i$*, then* $I_{ij} = \left[1 - v_{ij}^-, 1\right]$ *and* $S_{ij} = \{1 - v_{ij}^-\}$.

(d) *If* $a_{ij}^+ \geq b_i$ *and* $a_{ij}^- \geq b_i$*, then* $I_{ij} = \left[1 - v_{ij}^-, v_{ij}^+\right]$ *and* $S_{ij} = \begin{cases} \{1 - v_{ij}^-\} \bigcup \{v_{ij}^+\} & , v_{ij}^+ > 1 - v_{ij}^- \\ \left[1 - v_{ij}^-, v_{ij}^+\right] & , v_{ij}^+ \leq 1 - v_{ij}^- \end{cases}$.

Definition 2.5. For each $j \in \mathscr{J}$, we define $I^-(j) = \left\{i \in \mathscr{I} : a_{ij}^- \geq b_i\right\}$, $I^+(j) = \left\{i \in \mathscr{I} : a_{ij}^+ \geq b_i\right\}$ and $I_j = \bigcap_{i \in \mathscr{I}} I_{ij}$. As well, define $S'_{ij} = S_{ij} \cap I_j$, $\forall i \in \mathscr{I}$ and $\forall j \in \mathscr{J}$.

Remark 2.6. By Corollary 2.4 and Definition 2.5, it follows that $I_j = [L_j, U_j]$, $\forall j \in \mathscr{J}$, where

$$L_j = \begin{cases} \max_{i \in I^-(j)} \left\{1 - v_{ij}^-\right\} & , I^-(j) \neq \varnothing \\ 0 & , I^-(j) = \varnothing \end{cases} \quad , \quad U_j = \begin{cases} \min_{i \in I^+(j)} \left\{v_{ij}^+\right\} & , I^+(j) \neq \varnothing \\ 1 & , I^+(j) = \varnothing \end{cases}$$

Example 2.7. Considering Problem (1) with the Hamacher t-norm where:

$$A^+ = \begin{bmatrix} 0.24 & 0.06 & 1.00 & 0.53 & 0.00 & 0.65 \\ 0.85 & 0.48 & 0.06 & 0.63 & 0.70 & 0.25 \\ 0.35 & 0.23 & 0.05 & 0.58 & 0.00 & 0.80 \\ 0.00 & 0.82 & 0.30 & 0.55 & 0.89 & 0.65 \\ 0.13 & 0.10 & 0.12 & 0.46 & 0.11 & 0.04 \end{bmatrix}, \quad A^- = \begin{bmatrix} 0.08 & 0.20 & 0.41 & 0.06 & 0.83 & 0.78 \\ 0.35 & 0.51 & 0.25 & 0.35 & 0.78 & 0.04 \\ 0.53 & 0.03 & 0.14 & 0.00 & 0.01 & 1.00 \\ 0.81 & 0.00 & 0.32 & 0.72 & 0.34 & 0.36 \\ 0.15 & 0.35 & 0.21 & 0.28 & 0.40 & 0.33 \end{bmatrix}$$

$$b^T = \left[\ 0.8\ 0.5\ 0.9\ 0.4\ 0.3\ \right]$$

In this example, $\mathscr{I} = \{1, 2, \ldots, 5\}$, $\mathscr{J} = \{1, 2, \ldots, 6\}$ and we use the max-Hamacher t-norm T_H^α (that is continuous and strict) with $\alpha = 2$. So, we have $T_H^2(x, y) = \frac{xy}{2-(x+y-xy)}$. For $i = 1$ and $j = 5$, we have $a_{15}^+ = 0.00$, $a_{15}^- = 0.83$ and $b_1 = 0.8$. So, from Corollary 2.4 (c) we obtain $I_{15} = [1 - v_{15}^-, 1] = [0.03, 1]$ and $S_{15} = \{1 - v_{15}^-\} = \{0.03\}$. Tables 1 and 2 show all the sets I_{ij} and S_{ij} for each $i \in \mathscr{I}$ and $j \in \mathscr{J}$, respectively. Further, by Definition 2.5 and Tables 1 and 2, it can be easily calculated that $I_5 = \bigcap_{i \in \mathscr{I}} I_{i5} = [0.32, 0.48]$ and $S'_{15} = S_{15} \cap I_5 = \{0.3\} \cap [0.32, 0.48]$. Tables 3 and 4 show all the sets I_j and S'_{ij} for each $i \in \mathscr{I}$ and $j \in \mathscr{J}$, respectively.

3 Resolution of the feasible solution set

The following lemma presents two necessary conditions for the feasibility of the Problem (1).

Lemma 3.1. ***(a)*** *If* $S(A^+, A^-, b) \neq \varnothing$*, then* $I_j \neq \varnothing$*,* $\forall j \in \mathscr{J}$*.* ***(b)*** *If* $S(A^+, A^-, b) \neq \varnothing$*, then for each* $i \in \mathscr{I}$*, there exists at least one* $j_i \in \mathscr{J}$ *such that* $S'_{ij_i} \neq \varnothing$.

Proof. **(a)** Suppose that $x \in S(A^+, A^-, b)$ and $I_{j_0} = \varnothing$ for some $j_0 \in \mathscr{J}$. Hence, there exists $i_0 \in \mathscr{I}$ such that $x \notin I_{i_0 j_0}$ (Definition 2.5). So, from Definition 2.2, we have $\max\{T_H^\alpha\left(a_{i_0 j_0}^+, x_{j_0}\right)$, $T_H^\alpha\left(a_{i_0 j_0}^-, 1 - x_{j_0}\right)\} > b_{i_0}$, that implies $x \notin S_{i_0}$. So, from $S(A^+, A^-, b) = \bigcap_{i \in \mathscr{I}} S_i$, we obtain $x \notin S(A^+, A^-, b)$ that is a contradiction. **(b)** Assume that $x \in S(A^+, A^-, b)$ and there exists some $i_0 \in \mathscr{I}$ such that $S'_{i_0 j} = \varnothing$, $\forall j \in \mathscr{J}$. So, $x_j \notin S'_{i_0 j} = S_{i_0 j} \cap I_j$, $\forall j \in \mathscr{J}$ (Definition 2.5). Now, if $x_j \notin I_j$ for some $j \in \mathscr{J}$, then from Part (a) we obtain $S(A^+, A^-, b) \neq \varnothing$, that is a contradiction.

On the other hand, if $x_j \notin S'_{i_0 j}$ and $x_j \in I_j$, $\forall j \in \mathscr{J}$, then it is concluded that $x_j \notin S_{i_0 j}$, $\forall j \in \mathscr{J}$, that implies $\max\left\{T_H^{\alpha}\left(a_{i_0 j}^{+}, x_j\right), T_H^{\alpha}\left(a_{i_0 j}^{-}, 1-x_j\right)\right\} < b_{i_0}$, $\forall j \in \mathscr{J}$ (Definition 2.2). Therefore, $x \notin S_{i_0}$, that contradicts the assumption that $x \in S\left(A^{+}, A^{-}, b\right)$. ■

The following lemma provides a necessary and sufficient condition ensuring that a given $x \in [0,1]^n$ is feasible for the Problem (1).

Lemma 3.2. *Let $x \in S\left(A^{+}, A^{-}, b\right)$ if and only if the following statements hold true:*

- ***(I)*** $x_j \in I_j$, $\forall j \in \mathscr{J}$.
- ***(II)*** *For each $i \in \mathscr{I}$, there is at least one $j_i \in \mathscr{J}$ such that $x_{j_i} \in S'_{i j_j}$.*

Proof. Suppose that $x \in [0,1]^n$ satisfies the conditions (I) and (II). So, from the condition (I) and Definitions 2.2 and 2.5, we have $x_j \in I_{ij}$ and therefore $\max\left\{T_H^{\alpha}\left(a_{ij}^{+}, x_j\right), T_H^{\alpha}\left(a_{ij}^{-}, 1-x_j\right)\right\} \le b_i$, $\forall i \in \mathscr{I}$ and $\forall j \in \mathscr{J}$. On the other hand, from the condition (II), and Definitions 2.2 and 2.5, for each $i \in \mathscr{I}$ we have $\max\left\{T_H^{\alpha}\left(a_{ij_i}^{+}, x_{j_i}\right), T_H^{\alpha}\left(a_{ij_i}^{-}, 1-x_{j_i}\right)\right\} = b_i$ for some $j_i \in \mathscr{J}$. Consequently, for each $i \in \mathscr{I}$, it is concluded that $\max_{j=1}^{n}\left\{\max\left\{T_H^{\alpha}\left(a_{ij}^{+}, x_j\right), T_H^{\alpha}\left(a_{ij}^{-}, 1-x_j\right)\right\}\right\} = \max\left\{T_H^{\alpha}\left(a_{ij_i}^{+}, x_{j_i}\right), T_H^{\alpha}\left(a_{ij_i}^{-}, 1-x_{j_i}\right)\right\} = b_i$. Thus, $x \in S_i$, $\forall i \in \mathscr{I}$, that implies $x \in \bigcap_{i \in \mathscr{I}} S_i = S\left(A^{+}, A^{-}, b\right)$. The converse statement is obtained by reversing the argument. ■

Corollary 3.3. *For each $i \in \mathscr{I}$, $x \in S_i$ if and only if $x_j \in I_{ij}$, $\forall j \in \mathscr{J}$, and there is at least one $j_i \in \mathscr{J}$ such that $x_{j_i} \in S'_{ij_i}$.*

Definition 3.4. For each $i \in \mathscr{I}$, we define $\mathscr{J}_i = \left\{j \in \mathscr{J} : S'_{ij} \neq \varnothing\right\}$. Similarly, for each $j \in \mathscr{J}$, define $\mathscr{I}_j = \left\{i \in \mathscr{I} : S'_{ij} \neq \varnothing\right\}$.

Definition 3.5. A function e (on $\mathscr{I}$) will be considered admissible function if $e(i) \in \mathscr{J}_i(e)$, $\forall i \in \mathscr{I}$, where

- **(a)** $\mathscr{J}_1(e) = \mathscr{J}_1$.
- **(b)** $\mathscr{I}_j(e,i) = \{k \in \mathscr{I} : 1 \le k < i \text{ and } e(k) = j\}$, $\forall j \in \mathscr{J}$ and $\forall i \in \mathscr{I} - \{1\}$.
- **(c)** $\mathscr{J}_i(e) = \left\{j \in \mathscr{J}_i : \mathscr{I}_j(e,i) = \varnothing \text{ or } S'_{ij} \cap \left(\bigcap_{k \in \mathscr{I}_j(e,i)} S'_{kj}\right) \neq \varnothing\right\}$, $\forall i \in \mathscr{I} - \{1\}$.

Accordingly, let E be the set of all admissible functions. Thus, we can represent each e as the vector $e = [j_1, \ldots, j_m]$ in which $e(i) = j_i$, $\forall i \in \mathscr{I}$.

Definition 3.6. For each $e \in E$, let $\mathscr{I}_j(e) = \{i \in \mathscr{I} : e(i) = j\}$ and $S(e)$ be the set of all the vectors $x = (x_1, \ldots, x_n)$ such that

$$x_j \in \begin{cases} \bigcap_{i \in \mathscr{I}_j(e)} S'_{ij} & , \mathscr{I}_j(e) \neq \varnothing \\ I_j & , \mathscr{I}_j(e) = \varnothing \end{cases} \quad , \forall j \in \mathscr{J} \tag{4}$$

Corollary 3.7. *Let $e : \mathscr{I} \to \bigcup_{i \in \mathscr{I}} \mathscr{J}_i$ be a function so that $e(i) \in \mathscr{J}_i$, $\forall i \in \mathscr{I}$. Then, $e \in E$ if and only if $\bigcap_{i \in \mathscr{I}_j(e)} S'_{ij} \neq \varnothing$ for each $j \in \mathscr{J}$ such that $\mathscr{I}_j(e) \neq \varnothing$.*

Remark 3.8. By Corollary 3.7, the number of admissible functions is limited to $\prod_{i\in\mathscr{I}} |\mathscr{J}_i|$, where $|\mathscr{J}_i|$ indicates the cardinality of $\mathscr{J}_i$. In practice, the precise number of admissible functions is, in most cases, much less than this value.

The following theorem determines the feasible solution set of the Problem (1) by using the admissible functions.

Theorem 3.9. $S(A^+, A^-, b) = \bigcup_{e\in E} S(e)$.

Proof. Let $x \in \bigcup_{e\in E} S(e)$. So, $x \in S(e_0)$ for some $e_0 \in E$. Hence, according to (4), for each $j \in \mathscr{J}$ we have either $x_j \in \bigcap_{i\in\mathscr{I}_j(e_0)} S'_{ij}$ (if $\mathscr{I}_j(e_0) \neq \varnothing$) or $x_j \in I_j$ (if $\mathscr{I}_j(e_0) = \varnothing$). But, since $S'_{ij} = S_{ij} \cap I_j$ (Definition 2.5), from $x_j \in \bigcap_{i\in\mathscr{I}_j(e_0)} S'_{ij}$ it is obtained again $x_j \in I_j$. Consequently, $x_j \in I_j$, $\forall j \in \mathscr{J}$. On the other hand, from Definition 3.5 we have $e_0(i) = j_i \in \mathscr{J}_i(e_0) \subseteq \mathscr{J}_i$ ($\forall i \in \mathscr{I}$), which implies $\mathscr{I}_{j_i}(e_0) \neq \varnothing$, and therefore from (4) we have $x_{j_i} \in \bigcap_{k\in\mathscr{I}_{j_i}(e_0)} S'_{kj_i} \subseteq S'_{ij_i}$. Hence, $x_{j_i} \in S'_{ij_i}$, $\forall i \in \mathscr{I}$. Now, Lemma 3.2 requires that $x \in S(A^+, A^-, b)$. Conversely, let $x \in S(A^+, A^-, b)$, $\mathscr{I}_j(x) = \left\{i \in \mathscr{I} : x_j \in S'_{ij}\right\}$ and $\mathscr{J}_i(x) = \left\{j \in \mathscr{J} : x_j \in S'_{ij}\right\}$. So, for each $i \in \mathscr{I}_j(x)$ and each $j \in \mathscr{J}_i(x)$, we have $S'_{ij} \neq \varnothing$ that means $j \in \mathscr{J}_i$ (Definition 3.4). Also, Lemma 3.2 implies that $\mathscr{J}_i(x) \neq \varnothing$, $\forall i \in \mathscr{I}$. Without loss of generality, let $j_i = \min \mathscr{J}_i(x)$ and $e_0(i) = j_i$, $\forall i \in \mathscr{I}$. Therefore, e_0 is a function on $\mathscr{I}$ such that

$$e_0(i) = j_i \in \mathscr{J}_i,\ \forall i \in \mathscr{I} \tag{5}$$

Moreover, we have

$$\mathscr{I}_j(e_0) \neq \varnothing,\ \forall j \in \{j_1, \ldots, j_m\} \quad \text{and} \quad \mathscr{I}_j(e_0) = \varnothing,\ \forall j \in \mathscr{J} - \{j_1, \ldots, j_m\} \tag{6}$$

In addition, if $j_p \in \{j_1, \ldots, j_m\}$ and $k \in \mathscr{I}_{j_p}(e_0)$ (i.e., $e_0(k) = j_p$), then by our definition it follows that $j_p = \min \mathscr{J}_k(x)$ which means $x_{j_p} \in S'_{kj_p}$. Hence, it is concluded that

$$x_j \in \bigcap_{i\in\mathscr{I}_j(e_0)} S'_{ij},\ \forall j \in \{j_1, \ldots, j_m\} \tag{7}$$

Consequently, by Corollary 3.7 and (5) - (7) we have $e_0 \in E$. Also, since $x \in S(A^+, A^-, b)$, then $x_j \in I_j$, $\forall j \in \mathscr{J}$ (Lemma 3.2). Particularly, $x_j \in I_j$, if $\mathscr{I}_j(e_0) = \varnothing$. This fact together with (4), (6) and (7) imply $x \in S(e_0)$. ■

Example 3.10. Consider the problem stated in Example 2.7. According to Definition 3.4 and Table 4, we obtain $\mathscr{J}_1 = \{3\}$, $\mathscr{J}_2 = \{1, 5\}$, $\mathscr{J}_3 = \{6\}$, $\mathscr{J}_4 = \{1, 2, 4, 5, 6\}$ and $\mathscr{J}_5 = \{2, 4\}$. Hence, according to Remark 3.8, the number of admissible functions is bounded above by $\prod_{i\in\mathscr{I}} |\mathscr{J}_i| = 1 \times 2 \times 1 \times 5 \times 2 = 20$. Now, noting Definition 3.5 and Corollary 3.7, consider functions $e_1 = [3, 1, 6, 1, 4]$ and $e_2 = [3, 5, 6, 2, 4]$ from $\mathscr{I}$ to $\bigcup_{i\in\mathscr{I}} \mathscr{J}_i$ so that $e_p(i) \in \mathscr{J}_i$ for any $i \in \mathscr{I}$ and $p \in \{1, 2\}$. So, $e_1(2) = e_1(4) = 1$ and $\mathscr{I}_4(e_1, 4) = \{1\}$ (Definition 3.5 (b)). Also, $S'_{21} = \{0.62\}$ and $S'_{41} = \{0.46\}$ (see Table 4). Therefore, $\mathscr{I}_4(e_1, 4) \neq \varnothing$ and $S'_{21} \cap \left(\bigcap_{k\in\mathscr{I}_4(e_1,4)} S'_{k4}\right) = S'_{21} \cap S'_{41} = \varnothing$, which implies $1 \notin \mathscr{J}_4(e_1)$ (Definition 3.5 (c)). As a result, since $e_1(4) = 1 \notin \mathscr{J}_4(e_1)$, based on Definition 3.5 it follows that e_1 is not admissible. However, e_2 is indeed an admissible function and accordingly, the three conditions of Definition 3.5 hold for e_2. Hence, from Definition 3.6, $S(e_2)$ is calculated as the Cartesian product $S(e_2) = [0.46, 0.62] \times \{0.53\} \times \{0.80\} \times \{0.74\} \times \{0.31\} \times \{0.10\}$.

4 Simplification techniques

The purpose of this subsection is to describe some simplification techniques that can be used to accelerate the determination of the feasible region. In this section, $S_{\{i\}}(A^+, A^-, b)$ indicates the feasible region of the reduced problem obtained by removing the i'th equation from Problem (1).

Lemma 4.1. *Suppose that $S(A^+, A^-, b) \neq \varnothing$ and $i_0 \in \mathscr{I}$. If $x \in S_{\{i_0\}}(A^+, A^-, b)$, then $x_j \in I_j$, $\forall j \in \mathscr{J}$. Particularly, $x_j \in I_{i_0 j}$, $\forall j \in \mathscr{J}$.*

Proof. From Theorem 3.9, $S_{\{i_0\}}(A^+, A^-, b) = \bigcup_{e' \in E'} S(e')$, where E' is the set of all the restrictions of admissible functions $e \in E$ to $\mathscr{I} - \{i_0\}$. So, for each $x \in S_{\{i_0\}}(A^+, A^-, b)$, there exists at least one $e'_0 \in E'$ such that $x \in S(e'_0)$. Now, from (4), for each $j \in \mathscr{J}$ we have either $x_j \in \bigcap_{i \in \mathscr{I}_j(e'_0)} S'_{ij}$ or $x_j \in I_j$. But, in the former case, since $S'_{ij} = S_{ij} \cap I_j$ (Definition 2.5), we have again $x_j \in \bigcap_{i \in \mathscr{I}_j(e'_0)} S'_{ij} \subseteq I_j$. Consequently, $x_j \in I_j$ (and therefore $x_j \in I_{i_0 j}$ from Definition 2.5), $\forall j \in \mathscr{J}$. ∎

Corollary 4.2. *Suppose that $S(A^+, A^-, b) \neq \varnothing$ and there exist $i_0 \in \mathscr{I}$ and $j_0 \in \mathscr{J}$ such that $S'_{i_0 j_0} = I_{j_0}$. Then, the i_0'th equation is a redundant constraint and it can be deleted.*

Proof. Let $x \in S_{\{i_0\}}(A^+, A^-, b)$. From Lemma 4.1, $x_{j_0} \in I_{j_0}$ and $x_j \in I_{i_0 j}$, $\forall j \in \mathscr{J}$. But, since $S'_{i_0 j_0} = I_{j_0}$, then $x_{j_0} \in I_{j_0} = S'_{i_0 j_0}$, and therefore from Corollary 3.3 it follows that x satisfies the i_0'th equation. ∎

Corollary 4.3. *Suppose that $S(A^+, A^-, b) \neq \varnothing$ and there exists $j_0 \in \mathscr{J}$ such that $I_{j_0} = \{k\}$ is a singleton set. Then, $x_{j_0} = k$ for each feasible solution x. Also, the j_0'th column and any equation i_0 such that $k \in S'_{i_0 j_0}$ can be removed from the problem.*

Proof. For any $x \in S(A^+, A^-, b)$, Lemma 3.2 implies $x_{j_0} \in I_{j_0} = \{k\}$, i.e., $x_{j_0} = k$. Now, from Lemma 4.1, if $x \in S_{\{i_0\}}(A^+, A^-, b)$, then $x_j \in I_{i_0 j}$, $\forall j \in \mathscr{J}$ (*1). But, since $k \in S'_{i_0 j_0}$, then we have $S'_{i_0 j_0} \neq \varnothing$, that together with $S'_{i_0 j_0} = S_{i_0 j_0} \cap I_{j_0}$ and $I_{j_0} = \{k\}$ imply $S'_{i_0 j_0} = \{k\}$. Consequently, $x_{j_0} \in S'_{i_0 j_0}$ (*2). Hence, from (*1),(*2) and Corollary 3.3, it follows that x satisfies the i_0'th equation. ∎

Corollary 4.4. *Suppose that $S(A^+, A^-, b) \neq \varnothing$ and there exist $i_0 \in \mathscr{I}$ and $j_0 \in \mathscr{J}$ such that $\mathscr{J}_{i_0} = \{j_0\}$ and $S'_{i_0 j_0} = \{k\}$ are singleton sets. Then, $x_{j_0} = k$ for each feasible solution x. Also, the j_0'th column and any equation i such that $k \in S'_{i j_0}$ can be removed from the problem.*

Proof. If $x \in S(A^+, A^-, b)$, then from Lemma 3.2, there exists at least one $j \in \mathscr{J}$ such that $x_j \in S'_{i_0 j}$. But, since $\mathscr{J}_{i_0} = \{j_0\}$, we have necessarily $x_{j_0} \in S'_{i_0 j_0} = \{k\}$, that means $x_{j_0} = k$. Moreover, suppose that $i \in \mathscr{I}$ and $k \in S'_{i j_0}$. If $x \in S_{\{i\}}(A^+, A^-, b)$, then x satisfies the i_0'th equation, and therefore we must have $x_{j_0} = k \in S'_{i j_0}$. Also, from Lemma 4.1, $x_j \in I_{ij}$, $\forall j \in \mathscr{J}$. Hence, Corollary 3.3 implies that x also satisfies the i'th equation. ∎

Corollary 4.5. *Suppose that $S(A^+, A^-, b) \neq \varnothing$ and there exist $i, i_0 \in \mathscr{I}$ such that $S'_{ij} \subseteq S'_{i_0 j}$, $\forall j \in \mathscr{J}$. Then, the i_0'th equation is a redundant constraint and it can be deleted.*

Proof. If $x \in S_{\{i_0\}}(A^+, A^-, b)$, then by Lemma 4.1, we conclude that $x_j \in I_{i_0 j}$, $\forall j \in \mathscr{J}$ (*1). Also, from Lemma 3.2, there exists at least one $j_i \in \mathscr{J}$ such that $x_{j_i} \in S'_{i j_i}$. But, since $S'_{ij} \subseteq S'_{i_0 j}$, then we have $x_{j_i} \in S'_{i_0 j_i}$ (*2). So, (*1), (*2) and Corollary 3.3 imply that x satisfies the i_0'th equation. ∎

Corollary 4.6. *Suppose that $S(A^+, A^-, b) \neq \varnothing$ and $i_0 \in \mathscr{I}$. If $b_{i_0} = 0$, then the i_0'th equation is a redundant constraint and it can be deleted.*

Proof. Let $x \in S_{\{i_0\}}(A^+, A^-, b)$. We shall show that x also satisfies the i_0'th equation. From Lemma 4.1, $x_j \in I_j$, $\forall j \in \mathscr{J}$ (*1). Also, since $S(A^+, A^-, b) \neq \varnothing$, there exists at least one $j_0 \in \mathscr{J}$ such that $S'_{i_0 j_0} \neq \varnothing$ (Lemma 3.1 (b)). On the other hand, since $b_{i_0} = 0$, then we have $S'_{i_0 j_0} = I_{i_0 j_0}$ which together with $I_{j_0} \subseteq I_{i_0 j_0}$ (Definition 2.5) and (*1) imply $x_{j_0} \in S'_{i_0 j_0}$ (*2). Now, the result follows from (*1),(*2) and Corollary 3.3. ■

The following algorithm summarizes the preceding discussion.

Algorithm 1: Resolution of Problem (1)

Input : Given Problem (1):
Output: The optimal value x^*

1 Compute sets I_{ij}, S_{ij}, I_j and S'_{ij} for each $i \in \mathscr{I}$ and each $j \in \mathscr{J}$ (Corollary 2.4 and Definition 2.5).
2 **if** *$I_j \neq \varnothing$ for some $j \in \mathscr{J}$, **And** $S'_{ij} \neq \varnothing$ for some $i \in \mathscr{I}$ and each $j \in \mathscr{J}$* **then**
3 Determine as many variables as possible using the simplification rules. Remove the redundant equations and the corresponding columns from the problem.
4 Calculate $S(e)$ for each admissible function $e \in E$ of the remaining problem (Definitions 3.5 and 3.6). Assign the variable values found in Step 6 to $S(e)$.
5 Generate the feasible solution set $S(A^+, A^-, b)$ by $\bigcup_{e \in E} S(e)$ (Theorem 3.9).
6 Assign $x^*(e_i)$ to the objective function, and the minimum objective value is the optimal solution.
7 **end**
8 **else**
9 Stop; the problem is infeasible (Lemma 3.1).
10 **end**

5 Local and global optimal solutions

As mentioned before, by considering f is a linear and continuous function, that is non-decreasing (non-increasing) in x_j, $\forall j \in \mathscr{J}^+$ ($\forall j \in \mathscr{J}^-$).

Definition 5.1. Suppose that $S(A^+, A^-, b) \neq \varnothing$. For each $e \in E$, we define $x^*(e) = (x^*(e)_1, \ldots, x^*(e)_n)$ where for each $j \in \mathscr{J}$, components $x^*(e)_j$ are defined as follows:

$$x^*(e)_j = \begin{cases} \min\left\{\bigcap_{i \in \mathscr{I}_j(e)} S'_{ij}\right\} & , j \in \mathscr{J}^+ \text{ and } \mathscr{I}_j(e) \neq \varnothing \\ L_j & , j \in \mathscr{J}^+ \text{ and } \mathscr{I}_j(e) = \varnothing \\ \max\left\{\bigcap_{i \in \mathscr{I}_j(e)} S'_{ij}\right\} & , j \in \mathscr{J}^- \text{ and } \mathscr{I}_j(e) \neq \varnothing \\ U_j & , j \in \mathscr{J}^- \text{ and } \mathscr{I}_j(e) = \varnothing \end{cases} \tag{8}$$

where L_j and U_j are the lower and upper bounds of $I_j = [L_j, U_j]$. Also, define $F^* = \{x^*(e) : e \in E\}$.

The following theorem states that any $x^*(e)$ is a feasible local optimal solution of the problem (1).

Theorem 5.2. *Suppose that* $S(A^+, A^-, b) \neq \varnothing$.***(a)*** $F^* \subseteq S(A^+, A^-, b)$.***(b)*** $x^*(e)$ *is a global optimum in* $S(e)$

Proof. **(a)** Let $x^*(e_0) \in F^*$ (for some $e_0 \in E$). So, from (4) and (8), it follows that $x^*(e_0) \in S(e_0) \subseteq \bigcup_{e \in E} S(e)$. Hence, by Theorem 1, it is concluded that $F^* \subseteq S(A^+, A^-, b)$. **(b)** Let $x \in S(e)$. So, from (5) and (9), we have $x^*(e)_j \leq x_j$, $\forall j \in \mathscr{J}^+$, and $x_j \leq x^*(e)_j$, $\forall j \in \mathscr{J}^-$. Hence, $f(x^*(e)) \leq f(x)$ for any $x \in S(e)$. ■

Theorem 5.3. *Suppose that* $S(A^+, A^-, b) \neq \varnothing$ *and* S^* *denotes the set of optimal solutions for Problem (1). If* $f(x^*(e^*)) = \min\{x^*(e) : e \in E\}$*, then* $x^*(e^*)$ *is a global optimal solution of Problem (1).*

Proof. Let $x \in S(A^+, A^-, b)$ be an arbitrary feasible solution. We shall show that $f(x^*(e^*)) \leq f(x)$. From Theorem 3.9, $x \in S(e_0)$ for some $e_0 \in E$. So, theorem 5.2 requires $f(x^*(e_0)) \leq f(x)$, which together with $f(x^*(e^*)) \leq f(x^*(e_0))$ imply $f(x^*(e^*)) \leq f(x)$. ■

Based on 5.2, $x^*(e)$ is a local optimal solution, $\forall e \in E$. So, Theorem 5.3 further provides a necessary optimality condition stating that if x^* is an optimal solution of the Problem (1), then it must belong to F^*. Thus, each solution $x^*(e) \in F^*$ is an optimal candidate solution. As well, the global optimal value is calculated as $f(x^*(e^*)) = \min\{x^*(e) : e \in E\}$.

6 Numerical example

Consider the problems in Examples 2.7 and 3.10. Here are the steps according to Algorithm 1:

Step 1 Tables 1 and 2 show all the sets I_{ij} and S_{ij} for each $i \in \mathscr{I}$ and $j \in \mathscr{J}$, respectively. In both tables, row i ($i \in \mathscr{I}$) corresponds to equation i and column j ($j \in \mathscr{J}$) corresponds to variable x_j. By Table 1 and Definition 2.5, all the sets I_j are shown in Table 3 for each $j \in \mathscr{J}$. Also, all the sets S'_{ij} ($\forall i \in \mathscr{I}$ and $\forall j \in \mathscr{J}$) are summarized in Table 4 by using Table 2, Table 3 and Definition 2.5.

Table 1: Sets I_{ij} for each $i \in \mathscr{I}$ and $j \in \mathscr{J}$.

$[0, 1]$	$[0, 1]$	$[0, 0.80]$	$[0, 1]$	$[0.03, 1]$	$[0, 1]$
$[0, 0.62]$	$[0.10, 1]$	$[0, 1]$	$[0, 0.84]$	$[0.31, 0.76]$	$[0, 1]$
$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$	$[0.10, 1]$
$[0.46, 1]$	$[0, 0.53]$	$[0, 1]$	$[0.38, 0.79]$	$[0, 0.46]$	$[0, 0.68]$
$[0, 1]$	$[0.09, 1]$	$[0, 1]$	$[0, 0.74]$	$[0.17, 1]$	$[0.06, 1]$

Step 2 From Table 3, it is clear that $I_j \neq \varnothing$, $\forall j \in \mathscr{J}$.

Step 3 As it was described in Table 4, for each $i \in \mathscr{I}$ there exists at least one $j_i \in \mathscr{J}$ such that $S'_{ij_i} \neq \varnothing$.

Table 2: Sets S_{ij} for each $i \in \mathscr{I}$ and $j \in \mathscr{J}$.

$\varnothing$	$\varnothing$	$\{0.80\}$	$\varnothing$	$\{0.03\}$	$\varnothing$
$\{0.62\}$	$\{0.01\}$	$\varnothing$	$\{0.84\}$	$\{0.31, 0.76\}$	$\varnothing$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{0.10\}$
$\{0.46\}$	$\{0.53\}$	$\varnothing$	$\{0.38, 0.79\}$	$\{0.46\}$	$\{0.68\}$
$\varnothing$	$\{0.09\}$	$\varnothing$	$\{0.74\}$	$\{0.17\}$	$\{0.06\}$

Table 3: Sets $I_j = [L_j, U_j]$ for each $j \in \mathscr{J}$.

$[0.46, 0.62]$	$[0.09, 0.53]$	$[0, 0.80]$	$[0.38, 0.74]$	$[0.31, 0.46]$	$[0.10, 0.68]$

Table 4: Sets S'_{ij} for each $i \in \mathscr{I}$ and $j \in \mathscr{J}$.

$\varnothing$	$\varnothing$	$\{0.80\}$	$\varnothing$	$\varnothing$	$\varnothing$
$\{0.62\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{0.31\}$	$\varnothing$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{0.10\}$
$\{0.46\}$	$\{0.53\}$	$\varnothing$	$\{0.38\}$	$\{0.46\}$	$\{0.68\}$
$\varnothing$	$\{0.09\}$	$\varnothing$	$\{0.74\}$	$\varnothing$	$\varnothing$

Step 4 Since $\mathscr{J}_1 = \{3\}$ is a singleton set (see Table 3) and $0.80 \in I_3 = [0, 0.80]$ (see Table 4), Corollary 4.4 indicates that $x_3 = 0.80$ for each feasible solution x (particularly, $x_3^* = 0.80$ for each optimal solution x^*). Also, column 3 and both rows 2 can be removed from the problem. Although, by this simplification technique, the upper bound of the number of admissible functions is still 20 (Example 3.10) as $\prod_{i\in\mathscr{I}-(1)} |\mathscr{J}_i| = 2 \times 1 \times 4 \times 2 = 20$; that is $|E| \leq 20$. simillary, according to Table 4, it turns out that $\mathscr{J}_3 = \{6\}$ and $S'_{36} = \{0.10\}$ are singleton sets and $0.10 \in I_6 = [0.10, 0.68]$. Thus, x_6 is assigned to $x_6 = 0.10$, and also column 6 and rows 3 can be deleted by Corollary 4.4. Consequently, we have $\prod_{i\in\mathscr{I}-(1,3)} |\mathscr{J}_i| = 2 \times 4 \times 2 = 16$ and $|E| \leq 16$. So, after applying the above simplification rules, columns $\{3, 6\}$ and rows $\{1, 3\}$ are deleted and we obtain $x_3^* = 0.80$ and $x_6^* = 0.10$. Therefore, the reduced matrices $A^+$and A^-, and the right-hand-side vector b become

$$A^+ = \begin{bmatrix} 0.85 & 0.48 & 0.63 & 0.70 \\ 0.00 & 0.82 & 0.55 & 0.89 \\ 0.13 & 0.10 & 0.46 & 0.11 \end{bmatrix}, \; A^- = \begin{bmatrix} 0.35 & 0.51 & 0.35 & 0.78 \\ 0.81 & 0.00 & 0.72 & 0.34 \\ 0.15 & 0.35 & 0.28 & 0.40 \end{bmatrix}, \; b = \begin{bmatrix} 0.5 \\ 0.4 \\ 0.3 \end{bmatrix}$$

The current matrices $A^+$and $A^-$are equivalent to three rows (rows 2, 4 and 5 in the main problem) and four columns (columns 1,2,4 and 5 in the main problem). Furthermore, Table 4 is updated as follows:

Table 5: Sets S'_{ij} for each $i \in \mathscr{I}$ and $j \in \mathscr{J}$.

$\{0.62\}$	$\varnothing$	$\varnothing$	$\{0.31\}$
$\{0.46\}$	$\{0.53\}$	$\{0.38\}$	$\{0.46\}$
$\varnothing$	$\{0.09\}$	$\{0.74\}$	$\varnothing$

Step 5 and 6 Based on the results obtained in the previous step, we have $|E|= 16$. The admissible functions $e \in E$ are obtained from Table 5 as follow:
$e_1 = [1, 1, 2],\ e_2 = [1, 1, 3],\ e_3 = [1, 2, 2],\ e_4 = [1, 2, 3],\ e_5 = [1, 3, 2],\ e_6 = [1, 3, 3],\ e_7 = [1, 4, 2],\ e_8 = [1, 4, 3]$
$e_9 = [4, 1, 2],\ e_{10} = [4, 1, 3],\ e_{11} = [4, 2, 2],\ e_{12} = [4, 2, 3],\ e_{13} = [4, 3, 2],\ e_{14} = [4, 3, 3],\ e_[15] = [4, 4, 2],\ e_{16} = [4, 4, 3]$.

Therefore, by Definition 3.5 we determine that $e_{k\in\{4,5,7,8,9,10,12,13\}}$ is admissible, whereas the remainder is not. So, by Definition 3.6, and when we assign variables $x_3^* = 0.80$ and $x_6^* = 0.10$, their corresponding sets $S(e)$ are as follows:

$S(e_4) = \{0.62\} \times \{0.53\} \times \{0.80\} \times \{0.74\} \times [0, 1] \times \{0.10\}$
$S(e_5) = \{0.62\} \times \{0.09\} \times \{0.80\} \times \{0.38\} \times [0, 1] \times \{0.10\}$
$S(e_7) = \{0.62\} \times \{0.09\} \times \{0.80\} \times [0, 1] \times \{0.46\} \times \{0.10\}$
$S(e_8) = \{0.62\} \times [0, 1] \times \{0.80\} \times \{0.74\} \times \{0.46\} \times \{0.10\}$
$S(e_9) = \{0.46\} \times \{0.09\} \times \{0.80\} \times [0, 1] \times \{0.31\} \times \{0.10\}$
$S(e_{10}) = \{0.46\} \times [0, 1] \times \{0.80\} \times \{0.74\} \times \{0.31\} \times \{0.10\}$
$S(e_{12}) = [0, 1] \times \{0.53\} \times \{0.80\} \times \{0.74\} \times \{0.31\} \times \{0.10\}$
$S(e_{13}) = [0, 1] \times \{0.09\} \times \{0.80\} \times \{0.38\} \times \{0.31\} \times \{0.10\}$

Therefore, from Theorem 3.9, $S(A^+, A^-, b) = \bigcup_{i\in\{4,5,7,8,9,10,12,13\}} S(e_i)$.

Step 7 Assume that the objective function of Problem (1) is defined as the linear form $f(x) = \sum_{j=1}^{n} c_j x_j$, where $c_j \in \mathbb{R}$, $\forall j \in \mathscr{J}$. So, it is clear that $\mathscr{J}^+ = \{j \in \mathscr{J} : c_j \geq 0\}$ and $\mathscr{J}^- = \{j \in \mathscr{J} : c_j < 0\}$. For instance, consider the problem stated in Examples 2.7 and 3.10 with the following objective function:

$$f(x) = 3x_1 + 2x_2 - x_3 - 4x_4 + x_5 + 2x_6$$

In this example, we have $\mathscr{J}^+ = \{1, 2, 5, 6\}$ and $\mathscr{J}^- = \{3, 4\}$. To calculate $x^*(e_i)$, we obtain:

$x^*(e_4) = [0.62, 0.53, 0.8, 0.74, 0.31, 0.1]$
$x^*(e_5) = [0.62, 0.09, 0.8, 0.38, 0.31, 0.1]$
$x^*(e_7) = x^*(e_8) = [0.62, 0.09, 0.8, 0.74, 0.46, 0.1]$
$x^*(e_9) = x^*(e_{10}) = [0.46, 0.09, 0.8, 0.74, 0.31, 0.1]$
$x^*(e_{12}) = [0.46, 0.53, 0.8, 0.74, 0.31, 0.1]$
$x^*(e_{113}) = [0.46, 0.09, 0.8, 0.38, 0.31, 0.1]$

Also, the objective values of the above local optimal solutions are computed as

$f(x^*(e_4)) = 0.2$
$f(x^*(e_5)) = 0.32$
$f(x^*(e_7)) = -0.97$
$f(x^*(e_9)) = -1.6$
$f(x^*(e_{12})) = -0.28$
$f(x^*(e_{13})) = -1.16$

So, by Theorem 5.3, the global minimum solution of the problem is $x^*(e_9)$ with an optimal value of $f(x^*(e_9)) = -1.6$.

7 Conclusion

In this paper, an algorithm was introduced for finding a global optimal solution to linear problems under a system of bipolar fuzzy relation equation constraints defined by the max-Hamacher family of t-norms. The analytical properties of the bipolar FREs constraints were studied, and the feasible solution set of the problem was completely identified by a finite number of compact sets, which may not be connected. Two necessary feasibility conditions and a necessary and sufficient condition were also included to determine the feasibility of the problem. Moreover, it was proved that the problem has a finite number of local optimal solutions, and a global optimal solution is actually the local optimum with the smallest objective function value. In addition, some simplification rules have also been proposed to speed up the problem-solving procedure.

References

[1] E Sanchez, *Solutions in composite fuzzy relation equation, application to medical diagnosis in brouwerian logic. gupta, mm, saridis, gn, gaines, br, eds., fuzzy automata and decision process*, 1977. DOI: 10.1016/B978-1-4832-1450-4.50017-1

[2] M. Mizumoto and H.-J. Zimmermann, "Comparison of fuzzy reasoning methods," *Fuzzy sets and systems*, vol. 8, no. 3, pp. 253–283, 1982. DOI: 10.1016/s0165-0114(82)80004-3

[3] A. D. Nola, E. Sanchez, W. Pedrycz, and S. Sessa, *Fuzzy relation equations and their applications to knowledge engineering*, 1989. DOI: 10.1007/978-94-017-1650-5

[4] L. Chen and P. P. Wang, "Fuzzy relation equations (i): The general and specialized solving algorithms," *Soft Computing*, vol. 6, no. 6, pp. 428–435, 2002. DOI: 10.1007/s00500-001-0157-3

[5] A. Markovskii, "On the relation between equations with max-product composition and the covering problem," *Fuzzy Sets and Systems*, vol. 153, no. 2, pp. 261–273, 2005. DOI: 10.1016/j.fss.2005.02.010

[6] L. Chen and P. P. Wang, "Fuzzy relation equations (ii): The branch-point-solutions and the categorized minimal solutions," *Soft Computing*, vol. 11, no. 1, pp. 33–40, 2007. DOI: 10.1007/s00500-006-0050-1

[7] J.-L. Lin, Y.-K. Wu, and S.-M. Guu, "On fuzzy relational equations and the covering problem," *Information Sciences*, vol. 181, no. 14, pp. 2951–2963, 2011. DOI: 10.1016/j.ins.2011.03.004

[8] J.-L. Lin, "On the relation between fuzzy max-archimedean t-norm relational equations and the covering problem," *Fuzzy Sets and Systems*, vol. 160, no. 16, pp. 2328–2344, 2009. DOI: 10.1016/j.fss.2009.01.012

[9] D. Dubois and H. Prade, *Fundamentals of fuzzy sets*. Springer Science & Business Media, 2012, vol. 7. DOI: 10.1007/978-1-4615-4429-6

[10] C. Tan, W. Yi, and X. Chen, "Hesitant fuzzy hamacher aggregation operators for multicriteria decision making," *Applied Soft Computing*, vol. 26, pp. 325–349, 2015. DOI: 10.1016/j.asoc.2014.10.007

[11] X. Tang, C. Fu, D.-L. Xu, and S. Yang, "Analysis of fuzzy hamacher aggregation functions for uncertain multiple attribute decision making," *Information Sciences*, vol. 387, pp. 19–33, 2017. DOI: 10.1016/j.ins.2016.12.045

[12] L. Gál, R. Lovassy, I. J. Rudas, and L. T. Kóczy, "Learning the optimal parameter of the hamacher t-norm applied for fuzzy-rule-based model extraction," *Neural Computing and Applications*, vol. 24, pp. 133–142, 2014. DOI: 10.1007/s00521-013-1499-3

[13] D. Dubois and H. Prade, "An introduction to bipolar representations of information and preference," *International Journal of Intelligent Systems*, vol. 23, no. 8, pp. 866–877, 2008. DOI: 10.1002/int.20297

[14] D. Dubois and H. Prade, "An overview of the asymmetric bipolar representation of positive and negative information in possibility theory," *Fuzzy sets and Systems*, vol. 160, no. 10, pp. 1355–1366, 2009. DOI: 10.1016/j.fss.2008.11.006

[15] S. Freson, B. De Baets, and H. De Meyer, "Linear optimization with bipolar max–min constraints," *Information Sciences*, vol. 234, pp. 3–15, 2013. DOI: 10.1016/j.ins.2011.06.009

[16] P. Li and Q. Jin, "On the resolution of bipolar max-min equations," *Kybernetika*, vol. 52, no. 4, pp. 514–530, 2016. DOI: 10.14736/kyb-2016-4-0514

[17] M. E. Cornejo, D. Lobo, and J. Medina, "Bipolar fuzzy relation equations based on the product t-norm," in *2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, IEEE, 2017, pp. 1–6. DOI: 10.1109/fuzz-ieee.2017.8015691

[18] P. Li and Y. Liu, "Linear optimization with bipolar fuzzy relational equation constraints using the łukasiewicz triangular norm," *Soft Computing*, vol. 18, no. 7, pp. 1399–1404, 2014. DOI: 10.1007/s00500-013-1152-1

[19] C.-C. Liu, Y.-Y. Lur, and Y.-K. Wu, "Linear optimization of bipolar fuzzy relational equations with max-łukasiewicz composition," *Information Sciences*, vol. 360, pp. 149–162, 2016. DOI: 10.1016/j.ins.2016.04.041

[20] X.-P. Yang, "Resolution of bipolar fuzzy relation equations with max-łukasiewicz composition," *Fuzzy Sets and Systems*, vol. 397, pp. 41–60, 2020. DOI: 10.1016/j.fss.2019.08.005

[21] J. Zhou, Y. Yu, Y. Liu, and Y. Zhang, "Solving nonlinear optimization problems with bipolar fuzzy relational equation constraints," *Journal of Inequalities and Applications*, vol. 2016, no. 1, pp. 1–10, 2016. DOI: 10.1186/s13660-016-1056-6

[22] A. Ghodousian and M. S. Chopannavaz, "Solving linear optimization problems subject to bipolar fuzzy relational equalities defined with max-strict compositions," *Information Sciences*, vol. 650, p. 119 696, 2023. DOI: 10.1016/j.ins.2023.119696

[23] A. Ghodousian, M. S. Chopannavaz, and W. Pedrycz, "On the resolution and linear optimization problems subject to a system of bipolar fuzzy relational equalities defined with continuous archimedean t-norms," *Iranian Journal of Fuzzy Systems*, vol. 22, no. 4, pp. 137–160, 2025. DOI: 10.22111/ijfs.2025.50906.8994

[24] A. Ghodousian and M. S. Chopannavaz, "On the nonlinear programming problems subject to a system of generalized bipolar fuzzy relational equalities defined with continuous t-norms," *Iranian Journal of Fuzzy Systems*, vol. 22, no. 3, pp. 21–38, 2025. DOI: 10.22111/ijfs.2025.50433.8905