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Research Article

A Numerical Approach with Spectral Technique to Solve Fractional $\mathbb{PDE}$s of Third Order

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Abstract

Abstract: In this paper, we construct an approximate solution for an initial-boundary value problem involving a third order partial differential equation. Also, we present an approximate solution for the fractional case of the problem by using Mittag–Lefler function. Our method is based on spectral method for solving the related spectral problem. Finally, the proposed method is tested on some numerical examples.
Keywords: Spectral method, Initial-boundary value problem, Third order partial differential equations, Fractional partial differential equations.
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1 Introduction

Spectral methods are a type of techniques used in math and computing to solve certain equations. Its goal is to solve a differential equation by breaking the solution down into smaller parts called basis functions, and then figuring out the numbers that go with each part. Those can be used to solve ordinary ($\mathbb{ODE}$s), partial ($\mathbb{PDE}$s), fractional differential equations ($\mathbb{FDE}$s) and eigenvalue problems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Applying the methods for time-dependent $\mathbb{PDE}$s, the solution is typically written as a sum of basis functions with time-dependent coefficients; substituting this solution in the $\mathbb{PDE}$ yields a system of $\mathbb{ODE}$s which can be solved. Eigenvalue problems for $\mathbb{ODE}$s are similarly converted to matrix eigenvalue problems. For more review of this method see [12] for linear periodic and Non-periodic problems, [13] for hyperbolic problems and [14] for nonlinear

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thermostatic bi-metal problems.
Two important the methods are:

- Galerkin method: Test functions are $\chi_n = \phi_n$ and each ϕ_n satisfy the boundary condition $\mathbb{B}\phi_n(y) = 0$ [15, 16].
- Tau method: (Lanczos 1938) Test functions $\chi_n = \phi_n$ (most of) trial functions but ϕ_n do not satisfy the boundary conditions, the latter are enforced by an additional condition [17, 18].

In this paper, at first, we obtain the related spectral problem by separation of the variables. Then for the resulted spectral problem, we determine eigenvalues and eigenfunctions. Finally, by using Galerkin method and eigenfunctions (as test function), we obtain a system of ordinary differential equations for the unknown coefficients of the time variable [19, 20]. In final section, we will consider a time fractional $\mathbb{PDE}$ and choose its approximate solution by Mittage-Lefler functions.

2 Preleminieries

We recall the fractional derivative definition and some considerations on Mittage-Lefler functions are as follows [21]. For the arbitrary function $u(z)$ and for $0 < \sigma < 1$, the fractional derivative of $u(z)$ is

$$\mathbb{D}^{\sigma}u(z) = \frac{1}{\Gamma(1-\sigma)}\frac{d}{dz}\int_0^z \frac{u(\xi)}{(z-\xi)^{\sigma}}\,d\xi, \tag{1}$$

For the arbitrary σ, the Gamma function cannot be used for negative whole numbers with no imaginary part. To solve this, first take the whole number derivative and then the fractional derivative.

We consider the following modified Mittag–Lefler function:

$$\hbar_{\iota^*}(z) = \sum_{k=1}^{\infty}\frac{z^{k\iota^*-1}}{(k\iota^*-1)!} = \frac{z^{\iota^*-1}}{(\iota^*-1)!} + \frac{z^{2\iota^*-1}}{(2\iota^*-1)!} + \frac{z^{3\iota^*-1}}{(3\iota^*-1)!} + \dots, \tag{2}$$

which we will use the general form of it to solve $\mathbb{FDE}$. Function (2) as same as Taylor expansion for $\exp(\eta z)$ is invariant with respect to ordinary differentiation that means

$$\mathbb{D}^{(n\iota^*)}\hbar_{\iota^*}(z) = \hbar_{\iota}^{*}(z).$$

Hence, we consider it with parameter η, that mean:

$$u(z) = \hbar_{\iota}(z,\eta) = \sum_{k=1}^{\infty}\frac{\eta^k z^{k\iota^*-1}}{(k\iota^*-1)!}. \tag{3}$$

Clearly,

$$u^{(n)}(z) = \mathbb{D}^{(n)}\hbar_{\iota^*}(z,\eta) = \eta^n\hbar_{\iota^*}(z,\eta). \tag{4}$$

Thanks to the function, the ordinary $\mathbb{FDE}$s:

$$a_m u^{m/n}(z) + a_{m-1}u^{(m-1)/n}(z) + \dots + a_1 u^{1/n}(z) + a_0 u = 0, \tag{5}$$

can be solved. Hence, we can solve these $\mathbb{FDE}$s by charactristic equations as same as $\mathbb{ODE}$:

$$a_m\eta^m + a_{m-1}\eta^{m-1} + \cdots + a_1\eta + a_0 = 0, \tag{6}$$

regarding the roots of Eq. (6) by $\eta_1, \eta_2, \ldots, \eta_m$ then general solution of Eq. (5) is

$$\mathfrak{u}(z) = c_1\hbar_{\iota^*}(z, \eta_1) + c_2\hbar_{\iota^*}(z, \eta_2) + \cdots + c_m\hbar_{\iota^*}(z, \eta_m), \tag{7}$$

where $\iota^* = \frac{1}{n}$ is fractional step derivative with the fractional1 orders $\frac{m}{n}, \frac{m-1}{n}, \ldots, \frac{1}{n}$ [23].

3 Main results

3.1 Mathematical statement of problem

Consider the following initial-boundary value problem

$$\frac{\partial \mathfrak{u}}{\partial t} + \alpha\frac{\partial \mathfrak{u}}{\partial z} = \beta\frac{\partial^2 \mathfrak{u}}{\partial z^2} + \gamma\frac{\partial^3 \mathfrak{u}}{\partial z^3}, \qquad z \in \mathbb{J} =: (0,1), t > 0, \gamma \neq 0, \tag{8}$$

with boundary conditions

$$\left.\frac{\partial^k \mathfrak{u}}{\partial z^k}\right|_{z=0} = \left.\frac{\partial^k \mathfrak{u}}{\partial z^k}\right|_{z=1}, \qquad k = 0, 1, 2, \tag{9}$$

and initial condition

$$\mathfrak{u}(z, 0) = \varphi_0(z), \qquad z \in \bar{\mathbb{J}} =: [0, 1], \tag{10}$$

where α, β, γ are real constants and $\varphi_0(z)$ is real real continious function on $\bar{\mathbb{J}}$. In with some compatibility conditions for φ_0 to the following form

$$\varphi_0^{(k)}(0) = \varphi_0^{(k)}(1), \qquad k = 0, 1, 2. \tag{11}$$

Furthermore, suppose the unknown function $\mathfrak{u}$ satisfies in additional condition:

$$\mathfrak{u} \in C^{(3,1)}(\mathcal{E}) \cap C^{(2,1)}(\bar{\mathcal{E}}), \qquad \mathcal{E} = \mathbb{J} \times (0, \infty). \tag{12}$$

Remark 3.1. In the section 3.3, we shall counsider an initial-boundary value problem which consists of a fractional order differential equation as follows

$$\frac{\partial^\sigma \mathfrak{u}}{\partial t^\sigma} = \alpha\frac{\partial \mathfrak{u}}{\partial z} + \beta\frac{\partial^2 \mathfrak{u}}{\partial z^2} + \gamma\frac{\partial^3 \mathfrak{u}}{\partial z^3}, \qquad z \in \mathbb{J}, t > 0, \gamma \neq 0, \tag{13}$$

for $\sigma \in \mathbb{J}$, α, β, $\gamma \in \mathbb{R}$, with the following initial-boundary conditions

$$\left.\frac{\partial^k \mathfrak{u}}{\partial z^k}\right|_{z=0} = \left.\frac{\partial^k \mathfrak{u}}{\partial z^k}\right|_{z=1}, \qquad k = 0, 1, 2, t \geq 0, \tag{14}$$

and $\frac{\partial \mathfrak{u}}{\partial t}(z, 0) = \varphi_0(z)$, $z \in \bar{\mathbb{J}}$, where $\varphi_0(z)$ is real continious function on $\bar{\mathbb{J}}$.

3.2 Related spectral problem

Let the solution of the Eq. (8) be in the form $\mathfrak{u}(z,t) = \mathfrak{Z}(z)\mathfrak{T}(t)$, then the spectral problem will be in the following form

$$\begin{cases} \mathfrak{Z}''' + \frac{\beta}{\gamma}\mathfrak{Z}'' - \frac{\alpha}{\gamma}\mathfrak{Z}' - \lambda^3\mathfrak{Z} = 0 \\ \mathfrak{Z}_0^{(k)} = \mathfrak{Z}_1^{(k)} & k = 0,1,2. \end{cases} \tag{15}$$

Then the general solution is in the form of

$$\mathfrak{Z}(z) = \sum_{m=1}^{3} \mathcal{O}_m \exp(\vartheta_m(\lambda)z),$$

where unknown coefficients $\mathcal{O}_m$ are calculated from the following algebraic system. By applying boundary conditions (15) to this solution, we have the following algebraic system

$$\begin{cases} \mathcal{O}_1\left(1-\exp(\vartheta_1\lambda)\right) + \mathcal{O}_2\left(1-\exp(\vartheta_2\lambda)\right) + \mathcal{O}_3\left(1-\exp(\vartheta_3\lambda)\right) = 0, \\ \mathcal{O}_1\vartheta_1(\lambda)\left(1-\exp(\vartheta_1\lambda)\right) + \mathcal{O}_2\vartheta_2(\lambda)\left(1-\exp(\vartheta_2\lambda)\right) + \mathcal{O}_3\vartheta_3(\lambda)\left(1-\exp(\vartheta_3\lambda)\right) = 0, \\ \mathcal{O}_1\vartheta_1^2(\lambda)\left(1-\exp(\vartheta_1\lambda)\right) + \mathcal{O}_2\vartheta_2^2(\lambda)\left(1-\exp(\vartheta_2\lambda)\right) + \mathcal{O}_3\vartheta_3^2(\lambda)\left(1-\exp(\vartheta_3\lambda)\right) = 0, \end{cases} \tag{16}$$

this algebraic system can be solved by Kramer role and therefore the eigenfunctions will be as follows.

$$\mathfrak{Z}_{\iota^* p}(z) = \sum_{m=1}^{3} \Delta^{1m}(\lambda_{\iota^* p}) \exp(\vartheta_m(\lambda_{\iota^* p})z), \qquad p \in \mathbb{Z}, \iota^* = 1,2,3, \tag{17}$$

where Δ^{1m} are the minor determinants of the main determinant of the algebraic system (16) and the $\lambda_{\iota^* p}$ are the three complex roots of the main determinant. It is easy to see that these functions are orthogonal, that is

$$\begin{aligned} (\mathfrak{Z}_{\iota^* p}(z), \psi_{\iota'^* q}(z)) &= \int_0^1 \mathfrak{Z}_{\iota^* p}(z)\overline{\psi_{\iota'^* q}}\, dz \\ &= \delta_{\iota^*\iota'^* pq} = \begin{cases} 0, & p \neq q,\ \iota^* \neq \iota'^*, \\ 1, & p = q,\ \iota^* = \iota'^*. \end{cases} \end{aligned} \tag{18}$$

Applying the initial condition (10) yields

$$\varphi_0(z) = \sum_{\iota^*=1}^{3} \sum_{p\in\mathbb{Z}} \sigma_{\iota^* p}\mathfrak{Z}_{\iota^* p}(z), \tag{19}$$

where

$$\sigma_{\iota'^* q} = \int_0^1 \varphi_0(z)\overline{\psi_{\iota'^* q}(z)}\, dz, \qquad q \in \mathbb{Z},\ \iota'^* = 1,2,3.$$

Hence the form of final solution is

$$\mathfrak{u}(z,t) = \sum_{\iota'^*=1}^{3} \sum_{q=-N}^{N} \mathcal{O}_{\iota'^*_q}(t)\mathfrak{Z}_{\iota'^*_q}(z) + \varphi_0(z), \tag{20}$$

where the unknown cofficients $\mathcal{O}_{\iota_q^{\prime *}}$ with conditions $\mathcal{O}_{\iota_q^{\prime *}}(0) = 0$, should be calculated. Due to the introduction and various spectral methods [22, 23], the residual term $\hat{R}$ for the above approximate solution is

$$\begin{aligned}\hat{R} = &\sum_{\iota^{\prime *}=1}^{3}\sum_{q=-N}^{N} \mathcal{O}'_{\iota_q^{\prime *}}(t)\mathfrak{z}_{\iota_q^{\prime *}}(z)\\ &+ \alpha \sum_{\iota^{\prime *}=1}^{3}\sum_{q=-N}^{N} \mathcal{O}_{\iota_q^{\prime *}}(t)\mathfrak{z}'{\iota_q^{\prime *}}(z) + \alpha\varphi_0'(z)\\ &- \beta \sum_{\iota^{\prime *}=1}^{3}\sum_{q=-N}^{N} \mathcal{O}_{\iota_q^{\prime *}}(t)\mathfrak{z}''{\iota_q^{\prime *}}(z) - \beta\varphi_0''(z)\\ &- \gamma \sum_{\iota^{\prime *}=1}^{3}\sum_{q=-N}^{N} \mathcal{O}_{\iota_q^{\prime *}}(t)\mathfrak{z}'''_{\iota_q^{\prime *}}(z) - \gamma\varphi_0'''(z).\end{aligned} \tag{21}$$

Now according to the Galerkin method, we should have the inner product $(\hat{R}, \phi_q) = 0$, that is

$$\int_0^1 \hat{R}.\overline{\psi_{\iota^* p}(z)}\,dz = 0, \qquad \iota^* = 1, 2, 3,\ p = -N, \dots, N. \tag{22}$$

After some algebraic operations, the following ordinary differential equations system for unknowns $\mathcal{O}_{\iota^* p}(t)$ are resulted.

$$\mathcal{O}'_{\iota^* p}(t) - \gamma\lambda^3_{\iota^* p}\mathcal{O}_{\iota^* p}(t) + \int_0^1 \left[\alpha\varphi_0'(z) - \beta\varphi_0''(z)\gamma\varphi_0'''(z)\right]\overline{\psi_{\iota^* p}(z)}\,dz = 0, \tag{23}$$

for $\iota^* = 1, 2, 3$ and $p = -N, \dots, N$. Finaly, by calculating the functions $\mathcal{O}_{\iota^* p}(t)$ from the above differential equations system with initial conditions $\mathcal{O}_{\iota^* p}(0) = 0$ and substituting them in the solution (20), the final approximate solution (20) is obtained.

3.3 Related examples

In this section, we consider two examples. The first example is to construct an approximate solution for an initial-boundary value problem which consists a third order $\mathbb{PDE}$.The secend example is to construct an approximate solution for an initial-boundary value problem which consists of a time fractional order $\mathbb{PDE}$.

Example 3.2. We consider boundary value problem (8) with initial and boundary conditions (9)-(11). In particular, let the test functions are $\{z^n\}$ then the approximate solution is as follows

$$\mathfrak{u}(z, t) = \sum_{n=0}^{N} \mathcal{O}_n(t) z^n, \tag{24}$$

By applying the boundary conditions (9) on Eq. (24) we get

$$\mathcal{O}_0(\mathrm{t})\mathrm{z}^0 = \sum_{n=0}^{N} \mathcal{O}_n(\mathrm{t}) \Longrightarrow \sum_{n=1}^{N} \mathcal{O}_n(\mathrm{t}) = 0,$$
$$\mathcal{O}_1(\mathrm{t})\mathrm{z}^0 = \sum_{n=1}^{N} n\mathcal{O}_n(\mathrm{t}) \Longrightarrow \sum_{n=2}^{N} n\mathcal{O}_n(\mathrm{t}) = 0,$$
$$2\mathcal{O}_2(\mathrm{t})\mathrm{z}^0 = \sum_{n=2}^{N} n(n-1)\mathcal{O}_n(\mathrm{t}) \Longrightarrow \sum_{n=3}^{N} n(n-1)\mathcal{O}_n(\mathrm{t}) = 0.$$

So we have the following algebraic system for $\mathcal{O}_n(\mathrm{t})$,

$$\mathcal{O}_3(t) = -\tfrac{1}{3!}\sum_{n=4}^{N} n(n-1)\mathcal{O}_n(\mathrm{t}),$$
$$\mathcal{O}_2(\mathrm{t}) = -\tfrac{1}{2!}\sum_{n=3}^{N} n\mathcal{O}_n(\mathrm{t}) = -\tfrac{1}{2!}\sum_{n=4}^{N} n(n-1)\mathcal{O}_n(\mathrm{t}) - \tfrac{1}{2!}\sum_{n=4}^{N} n\mathcal{O}_n(\mathrm{t})$$
$$= \sum_{n=4}^{N}\left[\tfrac{n(n-1)}{4} - \tfrac{n}{2}\right]\mathcal{O}_n(\mathrm{t}) = \sum_{n=4}^{N} \tfrac{n(n-3)}{4}\mathcal{O}_n(\mathrm{t}),$$
$$\mathcal{O}_1(\mathrm{t}) = -\sum_{n=2}^{N} \mathcal{O}_n(\mathrm{t}) = -\sum_{n=4}^{N} \tfrac{n(n-3)}{4}\mathcal{O}_n(\mathrm{t})$$
$$+ \tfrac{1}{3!}\sum_{n=4}^{N} n(n-1)\mathcal{O}_n(\mathrm{t}) - \sum_{n=4}^{N} \mathcal{O}_n(\mathrm{t})$$
$$= -\tfrac{1}{12}\sum_{n=4}^{N}(n^2 - 7n + 12)\mathcal{O}_n(\mathrm{t}) = -\sum_{n=4}^{N} \tfrac{(n-3)(n-4)}{3\times 4}\mathcal{O}_n(\mathrm{t}).$$

By considering the initial condition $\mathfrak{u}(\mathrm{z},0) = \varphi_0(\mathrm{z})$ we have

$$\varphi_0(\mathrm{z}) = \sum_{k=0}^{m} \sigma_k \mathrm{z}^k.$$

Therefore, we obtain

$$\sum_{n=0}^{N} \mathcal{O}_n(0)\mathrm{z}^n = \sum_{n=0}^{m} \sigma_n \mathrm{z}^n, \qquad \mathcal{O}_n(0) = \begin{cases} \sigma_n & n = 0, 1, \cdots, m \\ o & n > m. \end{cases} \tag{25}$$

So the residual term is

$$\hat{R} = \sum_{n=0}^{N} \mathcal{O}'_n(\mathrm{t})\mathrm{z}^n + \alpha\sum_{n=1}^{N} n\mathcal{O}_n(\mathrm{t})\mathrm{z}^{n-1} - \beta\sum_{n=2}^{N} n(n-1)\mathcal{O}_n\mathrm{z}^{n-2}$$
$$- \gamma\sum_{n=3}^{N} n(n-1)(n-2)\mathcal{O}_n\mathrm{z}^{n-3},$$

consequently,

$$0 = \int_0^1 \hat{R}\mathrm{z}^p \,\mathrm{dz} = \int_0^1 \mathrm{z}^p \Bigg[\sum_{n=0}^{N} \mathcal{O}_n'(\mathrm{t})\mathrm{z}^n + \alpha \sum_{n=0}^{N-1} (n+1)\mathcal{O}_{n+1}(\mathrm{t})\mathrm{z}^n - \beta\alpha \sum_{n=0}^{N-2} (n+2)(n+1)\mathcal{O}_{n+2}(\mathrm{t})\mathrm{z}^n - \gamma \sum_{n=0}^{N-3} (n+3)(n+2)(n+1)\mathcal{O}_{n+3}(\mathrm{t})\mathrm{z}^n \Bigg] \mathrm{dz}.$$

By calculating the above integral ,the differential equations system with respect to $\mathcal{O}_n(\mathrm{t})$ is resulted in the form

$$\sum_{n=0}^{N-3} \big[\mathcal{O}_n'(\mathrm{t}) + \alpha(n+1)\mathcal{O}_{n+1}(\mathrm{t}) - \beta(n+2)(n+1)\mathcal{O}_{n+2}(\mathrm{t}) - \gamma(n+3)(n+2)(n+1)\mathcal{O}_{n+3}(\mathrm{t})\big] \tfrac{1}{n+1+p} + \sum_{N-2}^{N} \mathcal{O}_n'(\mathrm{t}) \tfrac{1}{n+1+p} + \alpha \sum_{N-2}^{N} (n+1)\mathcal{O}_{n+1}(\mathrm{t}) \tfrac{1}{n+1+p} - \beta N(N-1)\mathcal{O}_n(\mathrm{t}) \tfrac{1}{N-1+p} = 0,$$

$$\mathcal{O}_1(\mathrm{t}) = -\sum_{n=4}^{N} \tfrac{(n-3)(n-4)}{4}\mathcal{O}_n(\mathrm{t}),\ \mathcal{O}_2(\mathrm{t}) = \sum_{n=4}^{N} \tfrac{n(n-3)}{4}\mathcal{O}_n(\mathrm{t}),\ \mathcal{O}_3(\mathrm{t}) = -\tfrac{1}{3!} \sum_{n=4}^{N} n(n-1)\mathcal{O}_n(\mathrm{t}), \tag{26}$$

with initial conditions

$$\mathcal{O}_n(0) = \begin{cases} \sigma_n, & n = 0, 1, \ldots, m, \\ 0, & n > m. \end{cases} \tag{27}$$

Finally, by computing the coefficients $\mathcal{O}_n(t)$ and substituting in Eq. (24), the final approximate solution will be obtained.

Example 3.3. We consider the following fractional boundary value problem

$$\frac{\partial^\sigma \mathfrak{u}}{\partial \mathrm{t}^\sigma} = \alpha \frac{\partial \mathfrak{u}}{\partial \mathrm{z}} + \beta \frac{\partial^2 \mathfrak{u}}{\partial \mathrm{z}^2} + \gamma \frac{\partial^3 \mathfrak{u}}{\partial \mathrm{z}^3}, \qquad \mathrm{z} \in \mathbb{J},\ \mathrm{t} > 0,\ \gamma \neq 0, \tag{28}$$

and $0 < \sigma < 1$, with initial- boundary conditions:

$$\left.\frac{\partial^k \mathfrak{u}}{\partial \mathrm{z}^k}\right|_{\mathrm{z}=0} = \left.\frac{\partial^k \mathfrak{u}}{\partial \mathrm{z}^k}\right|_{\mathrm{z}=1}, \qquad k = 0, 1, 2,\ \sigma \geq 0, \tag{29}$$

and

$$\frac{\partial^\alpha \mathfrak{u}}{\partial \mathrm{t}^\sigma}(\mathrm{z}, 0) = \varphi_0(\mathrm{z}), \qquad \mathrm{z} \in \overline{\mathbb{J}}.$$

For simplicity, let $\alpha = 1, \beta = \gamma = 0$. Based on the relation (3) the approximate solution is as below

$$u(z,t) = \sum_{k=1}^{4} \mathcal{O}_k(z) \frac{t^{k\sigma-1}}{(k\sigma-1)!} = \mathcal{O}_1(x) \frac{t^{\sigma-1}}{(\sigma-1)!} + \mathcal{O}_2(z) \frac{z^{2\sigma-1}}{(2\sigma-1)!} + \mathcal{O}_3(z) \frac{z^{3\sigma-1}}{(3\sigma-1)!} + \mathcal{O}_4(z) \frac{z^{4\sigma-1}}{(4\sigma-1)!}. \tag{30}$$

Due to the Mittag–Leffler functions (3) and (4), fractional derivative of order σ with respect to t is as follows

$$\mathbb{D}^{\sigma} u(z,t) = \mathcal{O}_2(z) \frac{t^{\sigma-1}}{(\sigma-1)!} + \mathcal{O}_3(z) \frac{t^{2\sigma-1}}{(2\sigma-1)!} + \mathcal{O}_4(z) \frac{t^{3\sigma-1}}{(3\sigma-1)!}. \tag{31}$$

Differentiation with respect to spatial variable z yeilds

$$\frac{\partial u(z,t)}{\partial z} = \mathcal{O}_1'(z) \frac{t^{\sigma-1}}{(\sigma-1)!} + \mathcal{O}_2'(z) \frac{t^{2\sigma-1}}{(2\sigma-1)!} + \mathcal{O}_3'(z) \frac{t^{3\sigma-1}}{(3\sigma-1)!} + \mathcal{O}_4'(z) \frac{t^{4\sigma-1}}{(4\sigma-1)!}. \tag{32}$$

By substituting the above derivatives in Eq. (28) we have

$$\mathcal{O}_2(z) \frac{t^{\sigma-1}}{(\sigma-1)!} + \mathcal{O}_3(z) \frac{t^{2\sigma-1}}{(2\sigma-1)!} + \mathcal{O}_4(z) \frac{t^{3\sigma-1}}{(3\sigma-1)!} = \mathcal{O}_1'(z) \frac{t^{\sigma-1}}{(\sigma-1)!} + \mathcal{O}_2'(z) \frac{z^{2\sigma-1}}{(2\sigma-1)!} + \mathcal{O}_3'(z) \frac{t^{3\sigma-1}}{(3\sigma-1)!}. \tag{33}$$

From Eq. (33), we obtain $\mathcal{O}_1'(z) = \mathcal{O}_2(z)$, $\mathcal{O}_2'(z) = \mathcal{O}_3(z)$, $\mathcal{O}_3'(z) = \mathcal{O}_4(z)$. As a result we have

$$\mathcal{O}_1'''(z) = \mathcal{O}_4(z), \quad \mathcal{O}_2''(z) = \mathcal{O}_4(z), \quad \mathcal{O}_3'(z) = \mathcal{O}_4(z).$$

Hence $\mathcal{O}_1(z) = \frac{z^3}{3!}$, $\mathcal{O}_2(z) = \frac{z^2}{2!}$ and $\mathcal{O}_3(z) = z$. Thus the approximate solution is

$$u(z,t) = \frac{z^3}{3!} \frac{t^{\sigma-1}}{(\sigma-1)!} + \frac{z^2}{2!} \frac{t^{2\sigma-1}}{(2\sigma-1)!} + \frac{z}{1!} \frac{t^{3\sigma-1}}{(3\sigma-1)!} + \frac{t^{4\sigma-1}}{(4\sigma-1)!}. \tag{34}$$

For this problem the approximate solutions for several values of σ are shown in Fig. 1. Also, plot of the approximate solution $u(z, 0.5)$ for different values of σ is shown in Fig. 2.

4 Conclusion

Spectral method is a relevant method for solving spectral problems which are resulted from initial-boundary value problems. In fact, by applying separation of variables method for initial-boundary value problem (1)-(2), we obtained the spectral problem (10). Then for this problem, we obtained eigenvalues and eigenfunctions. After that we searched an Approximate solution as infinite series based on eigenfunctions and unknown coefficients $\mathcal{O}_n(t)$. Then by solving the system of ordinary differential equations (23), we obtained the unknown coefficients $\mathcal{O}_n(t)$. Finaly we considered an initial-boundary value problem for a time fractional partial differential equation which its approximate solution was constructed by modifed Mitage-Lefler function.

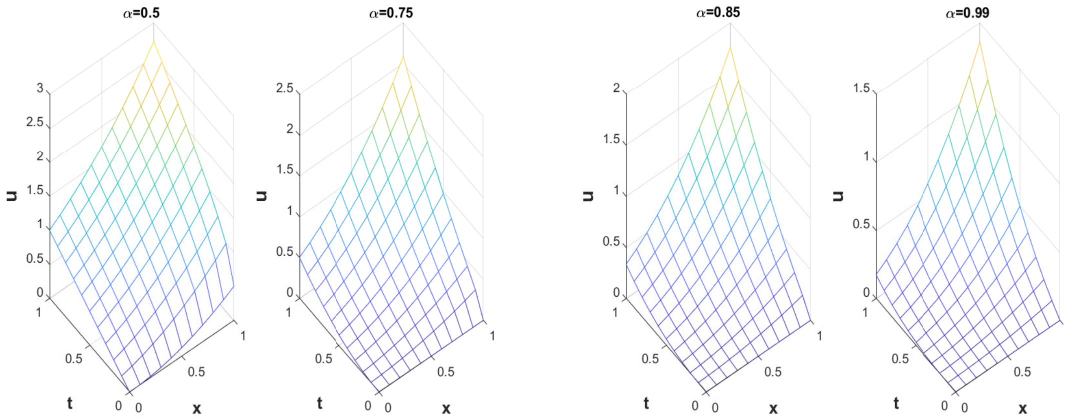

Figure 1: An image of the approximate solution for Example 3.2 with different values of σ.

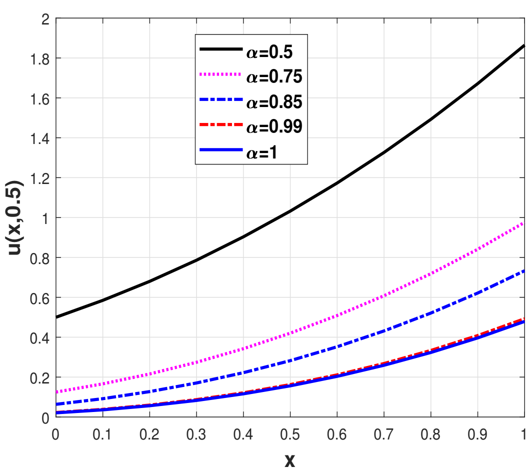

Figure 2: Plot of the approximate solution $u(z, 0.5)$ for different values of σ.

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