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Research Article

Fixed point theorems for $\alpha - \mu - 1 - 1-$upclass contractions in complete metric spaces

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Abstract

Abstract: In this paper, we extend the concept of contraction by presenting a new mapping called $\alpha - \mu - 1 - 1-$upclass contraction. We investigate the existence of fixed points for these mappings within complete metric spaces. Moreover, we furnish a demonstrative example to support our principal finding.

1 Introduction

Fixed point theory is a classical branch of nonlinear analysis. Its significance has grown rapidly over time, as it offers valuable tools for establishing the existence and uniqueness of solutions to various

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mathematical models, such as integral and partial differential equations, and variational inequalities. Additionally, it has a wide range of applications in fields such as engineering, economics, computer science, and many others.

It is widely recognized that contractive-type conditions are crucial in the study of fixed point theory, with the Banach contraction principle serving as a foundational element. Many researchers have extended and generalized this principle in diverse ways (see [1, 2, 3, 4, 5]), leading to numerous applications in mathematics and other related fields. In 2012, Samet *et al.* [6] presented fixed point theorems for mappings of α-ψ-contractive type, establishing theorems concerning fixed points within this context. In 2014, Popescu [7] delved into fixed point theorems for mappings falling under the generalized α-Geraghty type contraction in complete metric spaces. Subsequently, Ansari *et al.* [8] developed an extension of the Banach contraction principle by introducing a 1-1-upclass function, which has garnered significant attention from researchers. In 2019, Aydi *et al.* [9] introduced the concept of a fixed point result for α-β_E-Geraghty type contraction mappings, thereby extending the work of Fulga [10]. Recently, Bunpog *et al.* [11] extended the work of Aydi *et al.* by introducing α-β_M-Geraghty type contraction mappings, demonstrating both the existence and uniqueness of a fixed point for these mappings.

Inspired by [11, 8], we introduce a new concept of $\alpha-\mu-1-1-$upclass contraction to establish the existence and uniqueness of a fixed point in complete metric spaces. Moreover, we present examples to validate our findings.

2 Preliminaries

To begin, we provide some definitions and notations that will be utilized throughout this paper. We designate a nonempty set as X.

Definition 2.1. [12, 13] A mapping $d : X \times X \to [0, \infty)$ is called a metric on X if it satisfies the following conditions for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

Furthermore, the (X, d) is identified as a metric space.

Definition 2.2. [13] A sequence $\{x_n\}$ in a metric space (X, d) is Cauchy if for any $\varepsilon > 0$, there exists a positive integer N such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq N$, and converges to $x \in X$ if for any $\varepsilon > 0$, there exists a positive integer N such that $d(x_n, x) < \varepsilon$, for any $n \geq N$. We use the notation as $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and $\lim_{n\to\infty} d(x_n, x) = 0$, respectively. It is important to note that (X, d) is complete if every Cauchy sequence converges in X.

Lemma 2.3. *[14]Suppose (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$, $d(x_{m_k-1}, x_{n_k}) < \varepsilon$ and*

(i) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \varepsilon$;

(ii) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$;

(iii) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \varepsilon$.

We note that also can see $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \varepsilon$ *and* $\lim_{k\to\infty} d(x_{m_k}, x_{n_k-1}) = \varepsilon$

In 2012, Samet *et al.* [6] defined the concept of α-admissible. Later, Popescu [7] introduced the following concept:

Definition 2.4. [7] Let f be a self-mpping on X and let $\alpha : \mathrm{X} \times \mathrm{X} \to [0, +\infty)$ be a function, We say that f is a triangular α-orbital admissible if the following conditions hold:

(F1) $\alpha(x, fx) \geq 1 \implies \alpha(fx, f^2x) \geq 1$.

(F2) $\alpha(x, y) \geq 1 \wedge \alpha(y, fy) \geq 1 \implies \alpha(x, fy) \geq 1$, for all $x, y \in \mathrm{X}$.

To modify the notion of a α-admissible mapping, the following concept was introduced:

Definition 2.5. [15] Let f be a self-mpping on X and let $\mu : \mathrm{X} \times \mathrm{X} \to [0, +\infty)$ be a function, We say that f is a triangular μ-orbital subadmissible if the following conditions hold:

(M1) $\mu(x, y) \leq 1 \implies \mu(fx, fy) \leq 1$.

(M2) $\mu(x, z) \leq 1 \wedge \mu(z, y) \leq 1 \implies \mu(x, y) \leq 1$, for all $x, y, z \in \mathrm{X}$.

In 2014, Ansari et al[8] introduced the concept of $1-1-$upclass.

Definition 2.6. [8]A mapping $h : [0, \infty) \to [0, \infty)$ is an *A-class* function if

$$h(t) \geq t,$$

for all $t \geq 0$.

Example 2.7. [8]The following functions $h : [0, \infty) \to [0, \infty)$ are an *A-class* function:

(1) $h(t) = a^t - 1$, $a > 1$, $t \in [0, \infty)$;

(2) $h(t) = mt$, $m \geq 1$, $t \in [0, \infty)$.

Definition 2.8. [8] A mapping $\mathcal{F} : [0, \infty)^4 \to \mathbb{R}$ is a $1-1-$upclass function if the following conditions hold for all $u, v, s, t \in [0, \infty)$

1. $\mathcal{F}(1, 1, s, t)$ is continuous;
2. $0 \leq u \leq 1, v \geq 1 \Rightarrow \mathcal{F}(u, v, s, t) \leq \mathcal{F}(1, 1, s, t) \leq s$;
3. $\mathcal{F}(1, 1, s, t) = s \Rightarrow s = 0$ or $t = 0$.

Remark 2.9. [8]Note that $\mathcal{F}(1, 1, 0, 0) = 0$.

Example 2.10. [8] The following functions $\mathcal{F} : [0, \infty)^4 \to \mathbb{R}$ are a $1-1-$upclass function for all $u, v, s, t \in [0, \infty)$:

1. $\mathcal{F}(u,v,s,t) = us - vt$, $\mathcal{F}(1,1,s,t) = s \Rightarrow t = 0$;
2. $\mathcal{F}(u,v,s,t) = \dfrac{us - vt}{1+vt}$, $\mathcal{F}(1,1,s,t) = s \Rightarrow t = 0$;
3. $\mathcal{F}(u,v,s,t) = \dfrac{us}{1+vt}$, $f(1,1,s,t) = s \Rightarrow s = 0$ or $t = 0$;
4. $\mathcal{F}(u,v,s,t) = \log_a \dfrac{ut + a^{us}}{1+vt}$, $a > 1$, $\mathcal{F}(1,1,s,t) = s \Rightarrow s = 0$ or $t = 0$;
5. $\mathcal{F}(u,v,s,t) = \ln \dfrac{u + e^{us}}{1+v}$,$\mathcal{F}(1,1,s,1) = s \Rightarrow s = 0$;
6. $\mathcal{F}(u,v,s,t) = (us + a)^{\frac{1}{1+vt}} - a$, $a > 1$, $\mathcal{F}(1,1,s,t) = s \Rightarrow t = 0$;
7. $\mathcal{F}(u,v,s,t) = us \log_{a+vt} a$, $a > 1$, $\mathcal{F}(1,1,s,t) = s \Rightarrow s = 0$ or $t = 0$

Definition 2.11. [16] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.12. [17] A functions $\varphi : [0,\infty) \to [0,\infty)$ is called an Ultra-altering distance function if the following properties are satisfied:

(i) (i) φ is continuous,

(ii) (ii) $\varphi(t) > 0$, for all $t > 0$ and $\varphi(0) \geq 0$.

Definition 2.13. [8]Let (X,d) denote a metric space, with $\alpha, \beta : X \times X \to [0,\infty)$, and $f : X \to X$ representing a mapping on X. If we have that

$$h(\psi(d(fx,fy))) \leq \mathcal{F}(\mu(x,y), \alpha(x,y), \psi(d(x,y)), \varphi(d(x,y))), \tag{1}$$

for all $x,y \in X$, where ψ, φ are the earlier described altering distance function, $\mathcal{F}$ is a $1-1-$upclass and h is an *A-class* function. Then, f is (CAB)-contractive mapping.

3 Main results

In this section, we aim to expand the scope of the results presented in the publication by [11] to encompass a broader set of mappings.

Definition 3.1. Let (X,d) be a metric space, with $\alpha, \mu : X \times X \to [0,\infty)$, and $f : X \to X$ representing a mapping on X. If we have that

$$h(\psi(d(fx,fy))) \leq \mathcal{F}(\mu(x,y), \alpha(x,y), \psi(M(x,y)), \varphi(M(x,y))), \tag{2}$$

for all $x, y \in X$, where ψ, φ are an altering distance function (or φ is an Ultra-altering distance function), $\mathcal{F}$ is a $1-1-$upclass and h is an *A-class* function. And

$$M(x,y) = \max \left\{ \begin{array}{c} d(x,y) + |d(x,fx) - d(y,fy)|, \\ d(x,fx) + |d(x,y) - d(y,fy)|, \\ d(y,fy) + |d(x,y) - d(x,fx)| \end{array} \right\}.$$

Then, f is $\alpha - \mu - 1 - 1-$upclass contraction.

Our initial primary theorem provides a sufficient condition for the existence of a fixed point for the aforementioned mappings within a metric space.

Theorem 3.2. *Let (X,d) be a complete metric space, with $\alpha, \mu : X \times X \to [0,\infty)$, and a mapping $f : X \to X$. We assume that f satisfies the following conditions:*

(i) f is $\alpha - \mu - 1 - 1-$upclass contraction;

(ii) f is triangular α-orbital admissible and triangular μ-orbital subadmissible;

(iii) $\alpha(x_0, fx_0) \geq 1$ and $\mu(x_0, fx_0) \leq 1$ for some $x_0 \in X$;

(iv) f is continuous.

Consequently, the set $Fix(f)$ is nonempty, and the sequence $\{f^n x_0\}$ converges to $\omega \in Fix(f)$.

Proof. Assuming condition (iii), Let x_0 be an arbitrary point in X satisfying $\alpha(x_0, fx_0) \geq 1$ and $\mu(x_0, fx_0) \leq 1$. We define a sequence x_n in a metric space (X,d,s) as $x_n = fx_{n-1} = f^n x_0$ for all $n \geq 1$. If there exists a nonnegative real number n such that $x_n = x_{n+1} = fx_n$, the proof is concluded. Throughout the proof, we assume $x_n \neq x_{n+1}$ for any nonnegative real number n.

It is known that $\alpha(x_0, x_1) = \alpha(x_0, fx_0) \geq 1$, $\mu(x_0, x_1) = \mu(x_0, fx_0) \leq 1$ and by condition (ii), we can deduce that $\alpha(x_n, x_{n+1}) = \alpha(f^n x_0, f^{n+1} x_0) \geq 1$, $\mu(x_n, x_{n+1}) = \mu(f^n x_0, f^{n+1} x_0) \leq 1$ for all $n \geq 0$. This process can be repeated to establish the inequality

$$\text{if} \quad \alpha(x_n, x_{n+1}) \geq 1 \text{ and } \alpha(x_{n+1}, fx_{n+1}) \geq 1 \quad \text{then} \quad \alpha(x_n, x_{n+2}) \geq 1.$$

And,

$$\text{if} \quad \mu(x_n, x_{n+1}) \leq 1 \text{ and } \mu(x_{n+1}, fx_{n+1}) \leq 1 \quad \text{then} \quad \mu(x_n, x_{n+2}) \leq 1.$$

Using induction, it can be deduced that

$$\alpha(x_n, x_m) \geq 1, \quad \text{for any } m \geq n \geq 0.$$

And,

$$\mu(x_n, x_m) \leq 1, \quad \text{for any } m \geq n \geq 0.$$

From Equation (2), it follows that

$$\begin{aligned} \psi(d(fx_{n-1}, fx_n)) &\leq h(\psi(d(fx_{n-1}, fx_n))), \\ &\leq \mathcal{F}(\mu(x_{n-1}, x_n), \alpha(x_{n-1}, x_n), \psi(M(x_{n-1}, x_n)), \varphi(M(x_{n-1}, x_n))), \\ &\leq \mathcal{F}(1, 1, \psi(M(x_{n-1}, x_n)), \varphi(M(x_{n-1}, x_n))) \leq \psi(M(x_{n-1}, x_n)). \end{aligned}$$

Hence, we have that for all $n \geq 1$,

$$\begin{aligned} 0 &< \psi(d(x_n, x_{n+1})) = \psi(d(fx_{n-1}, fx_n)), \\ &\leq \mathcal{F}(1, 1, \psi(M(x_{n-1}, x_n)), \varphi(M(x_{n-1}, x_n))) \leq \psi(M(x_{n-1}, x_n)). \end{aligned} \tag{3}$$

Note that

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n) + |d(x_{n-1}, x_n) - d(x_n, x_{n+1})|, d(x_n, x_{n+1})\}.$$

Suppose there exists an integer $n > 0$ such that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$.
By employing equation (3), we can obtain the subsequent inequality,

$$\mathcal{F}(1, 1, \psi(d(x_n, x_{n+1})), \varphi(d(x_n, x_{n+1}))) = \psi(d(x_n, x_{n+1})),$$

Since $\mathcal{F}$ is a $1-1-$upclass function, then $\psi(d(x_n, x_{n+1})) = 0$ or $\varphi(d(x_n, x_{n+1}))$.
As ψ, φ are altering function (or φ is an Ultra-altering function), then $d(x_n, x_{n+1}) = 0$, this leads to a contradiction. So, we conclude that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$, for any $n > 0$. Consequently, it follows that

$$M(x_{n-1}, x_n) = 2d(x_{n-1}, x_n) - d(x_n, x_{n+1}), \quad \text{for any } n \geq 1.$$

Since the sequence $\{d(x_{n-1}, x_n)\}$ is decreasing and bounded below by 0, there exists a value $t \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1}, x_n) = t$. We assume that $t > 0$. By letting $n \to \infty$ in (3), we derive

$$\begin{aligned} \psi(t) &= \psi(\lim_{n\to\infty} d(x_n, x_{n+1})) \leq \lim_{n\to\infty} \mathcal{F}(1, 1, \psi(M(x_{n-1}, x_n)), \varphi(M(x_{n-1}, x_n))), \\ &\leq \mathcal{F}(1, 1, \psi(t), \varphi(t)), \\ &\leq \psi(t). \end{aligned}$$

This yields

$$\mathcal{F}(1, 1, \psi(t), \varphi(t)) = \psi(t),$$

and therefore, we can conclude that , $\psi(t) = 0$ or $\varphi(t) = 0$, that is,

$$t = \lim_{n\to\infty} M(x_{n-1}, x_n) = 0.$$

This results in a contradiction, thus allowing us to conclude that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{4}$$

To establish the Cauchy property of the sequence x_n, we will employ a proof by contradiction. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then,by Lemma 2.3 there exists an $\varepsilon > 0$ and two subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ with $m_k > n_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \text{ and } d(x_{m_k-1}, x_{n_k}) < \varepsilon, \tag{5}$$

and

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k}) = \lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \varepsilon. \tag{6}$$

By virtue of the inequality $\alpha(x_{n_i-1}, x_{m_i-1}) \geq 1$ and $\mu(x_{n_i-1}, x_{m_i-1}) \leq 1$, from equations (2) and (5), we can deduce the following:

$$\begin{aligned}\psi(d(x_{n_i}, x_{m_i})) &\leq h(\psi(d(x_{n_i}, x_{m_i}))),\\ &\leq \mathcal{F}(\mu(x_{n_i-1}, x_{m_i-1}), \alpha(x_{n_i-1}, x_{m_i-1}), \psi(M(x_{n_i-1}, x_{m_i-1})), \varphi(M(x_{n_i-1}, x_{m_i-1}))),\\ &\leq \mathcal{F}(1, 1, \psi(M(x_{n_i-1}, x_{m_i-1})), \varphi(M(x_{n_i-1}, x_{m_i-1}))).\end{aligned}$$

Hence, we get

$$\psi(d(x_{n_i}, x_{m_i})) \leq \mathcal{F}(1, 1, \psi(M(x_{n_i-1}, x_{m_i-1})), \varphi(M(x_{n_i-1}, x_{m_i-1}))) \leq \psi(M(x_{n_i-1}, x_{m_i-1})). \tag{7}$$

where

$$M(x_{n_i-1}, x_{m_i-1}) = \max \left\{ \begin{matrix} d(x_{n_i-1}, x_{m_i-1}) + |d(x_{n_i-1}, x_{n_i}) - d(x_{m_i-1}, x_{m_i})|, \\ d(x_{n_i-1}, x_{n_i}) + |d(x_{n_i-1}, x_{m_i-1}) - d(x_{m_i-1}, x_{m_i})|, \\ d(x_{m_i-1}, x_{m_i}) + |d(x_{n_i-1}, x_{m_i-1}) - d(x_{n_i-1}, x_{n_i})| \end{matrix} \right\}.$$

From (4) and (6), we have

$$\lim_{i\to\infty} M(x_{n_i-1}, x_{m_i-1}) = \varepsilon. \tag{8}$$

By utilizing the equation (7), (6) and (8), this implies that

$$\psi(\varepsilon) \leq \mathcal{F}(1, 1\psi(\varepsilon), \varphi(\varepsilon)) \leq \psi(\varepsilon).$$

we can conclude that, $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, that is $\varepsilon = 0$, which is a contradiction. Therefore, we can conclude that the sequence x_n forms a Cauchy sequence. Due to the completeness of the metric space, there exists an element $\omega \in X$ such that

$$\lim_{n\to\infty} d(x_n, \omega) = 0.$$

Since f is continuous, we have $\omega = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f(\lim_{n\to\infty} x_n) = f\omega$, that is, $\omega \in Fix(f)$. Since $x_n = f^n x_0$, we can conclude $\{f^n x_0\}$ converges to ω. ■

Corollary 3.3. *Theorem 3.2 is valid when assumption (a) is substituted by one of the following statements:*

(a) *For all $x, y \in X$,*

$$h(\psi(d(fx, fy))) \leq \mu(x, y)\psi(M(x, y)) - \alpha(x, y)\varphi(M(x, y)),$$

when

$$M(x, y) = \max \left\{ \begin{matrix} d(x, y) + |d(x, fx) - d(y, fy)|, \\ d(x, fx) + |d(x, y) - d(y, fy)|, \\ d(y, fy) + |d(x, y) - d(x, fx)| \end{matrix} \right\}.$$

(b) *For all $x, y \in X$,*

$$h(\psi(d(fx, fy))) \leq \frac{\mu(x, y)\psi(M(x, y))}{1 + \alpha(x, y)\varphi(M(x, y))},$$

when

$$M(x, y) = \max \left\{ \begin{matrix} d(x, y) + |d(x, fx) - d(y, fy)|, \\ d(x, fx) + |d(x, y) - d(y, fy)|, \\ d(y, fy) + |d(x, y) - d(x, fx)| \end{matrix} \right\}.$$

(*c*) *For all* $x, y \in X$ *and* $a > 1$,

$$h(\psi(d(fx, fy))) \leq \log_a \frac{\mu(x,y)\varphi(M(x,y)) + a^{\mu(x,y)\psi(M(x,y))}}{1 + \alpha(x,y)\varphi(M(x,y))},$$

when

$$M(x,y) = \max\left\{\begin{array}{c} d(x,y) + |d(x,fx) - d(y,fy)|, \\ d(x,fx) + |d(x,y) - d(y,fy)|, \\ d(y,fy) + |d(x,y) - d(x,fx)| \end{array}\right\}.$$

(*d*) *For all* $x, y \in X$ *and* $a > 1$,

$$h(\psi(d(fx, fy))) \leq (\mu(x,y)\psi(M(x,y)) + a)^{\dfrac{1}{1 + \alpha(x,y)\varphi(M(x,y))}},$$

when

$$M(x,y) = \max\left\{\begin{array}{c} d(x,y) + |d(x,fx) - d(y,fy)|, \\ d(x,fx) + |d(x,y) - d(y,fy)|, \\ d(y,fy) + |d(x,y) - d(x,fx)| \end{array}\right\}.$$

Proof.

(*a*) Apply $\mathcal{F}(u, v, s, t) = us - vt$ from Example 2.10 to Theorem 3.2.

(*b*) Apply $\mathcal{F}(u, v, s, t) = \dfrac{us}{1 + vt}$ from Example 2.10 to Theorem 3.2.

(*c*) Apply $\mathcal{F}(u, v, s, t) = \log_a \dfrac{ut + a^{us}}{1 + vt}$ from Example 2.10 to Theorem 3.2.

(*d*) Apply $\mathcal{F}(u, v, s, t) = (us + a)^{\dfrac{1}{1 + vt}}$ from Example 2.10 to Theorem 3.2. ■

Following that, we move on to introduce our second principal theorem. We replace the continuity requirement of the mapping f from Theorem 3.2 with an alternative criterion.

Theorem 3.4. *Let* (X, d) *be a complete metric space, with* $\alpha, \mu : X \times X \to [0, \infty)$, *and a mapping* $f : X \to X$. *We assume that* f *satisfies the following conditions:*

(i) f *is* $\alpha - \mu - 1 - 1-$*upclass contraction;*

(ii) f *is triangular* α*-orbital admissible and triangular* μ*-orbital subadmissible;*

(iii) $\alpha(x_0, fx_0) \geq 1$ *and* $\mu(x_0, fx_0) \leq 1$ *for some* $x_0 \in X$;

(iv) if a sequence $\{x_n\}$ *convergences to* $x \in X$ *and satisfies the condition* $\alpha(x_n, x_{n+1}) \geq 1$ *and* $\mu(x_n, x_{n+1}) \leq 1$ *for all* n, *then the existence of a subsequence* $\{x_{n_i}\}$ *from* $\{x_n\}$ *is guaranteed, which satisfies* $\alpha(x_{n_i}, x) \geq 1$ *and* $\mu(x_{n_i}, x) \leq 1$ *for all* i.

Consequently, the set $Fix(f)$ *is nonempty, and the sequence* $\{f^n x_0\}$ *converges to* $\omega \in Fix(f)$.

Proof. From the assertions outlined in Theorem 3.2, we can infer that the sequence, denoted as $x_n = f^n x_0$, converges to a limit point $\omega \in X$. By virtue of condition (iv), there exists a subsequence, denoted as x_{n_i} of x_n, such that $\alpha(x_{n_i}, \omega) \geq 1, \mu(x_{n_i}, \omega) \leq 1$ for all i. Moreover, based on condition (i), it can be established that

$$\begin{aligned}\psi(d(x_{n_i+1}, f\omega)) &\leq h(\psi(d(x_{n_i+1}, f\omega))),\\ &\leq \mathcal{F}(\mu(x_{n_i}, \omega), \alpha(x_{n_i}, \omega), \psi(M(x_{n_i}, \omega)), \varphi(M(x_{n_i}, \omega))),\\ &\leq \mathcal{F}(1, 1, \psi(M(x_{n_i}, \omega)), \varphi(M(x_{n_i}, \omega))) \leq \psi(M(x_{n_i}, \omega)).\end{aligned}$$

Hence, we have that for all $n \geq 1$,

$$\begin{aligned}0 &< \psi(d(x_{n_i+1}, f\omega))\\ &\leq \mathcal{F}(1, 1, \psi(M(x_{n_i}, \omega)), \varphi(M(x_{n_i}, \omega))) \leq \psi(M(x_{n_i}, \omega)).\end{aligned} \tag{9}$$

where

$$M(x_{n_i}, \omega) = \max\begin{Bmatrix} d(x_{n_i}, \omega) + |d(x_{n_i}, x_{n_i+1}) - d(\omega, f\omega)|, \\ d(x_{n_i}, x_{n_i+1}) + |d(x_{n_i}, \omega) - d(\omega, f\omega)|, \\ d(\omega, f\omega) + |d(x_{n_i}, \omega) - d(x_{n_i}, x_{n_i+1})| \end{Bmatrix}.$$

Assuming that $d(\omega, f\omega) > 0$, by employing the triangle inequality and (9), we derive the following for all i:

$$\begin{aligned}\psi(d(\omega, f\omega) - d(\omega, x_{n_i+1})) &\leq \psi(d(x_{n_i+1}, f\omega))\\ &\leq \mathcal{F}(1, 1, \psi(M(x_{n_i}, \omega)), \varphi(M(x_{n_i}, \omega)))\\ &\leq \psi(M(x_{n_i}, \omega)).\end{aligned}$$

Taking limit $i \to \infty$, we obtain

$$\psi(d(\omega, f\omega)) \leq \mathcal{F}(1, 1, \psi(d(\omega, f\omega)), \varphi(d(\omega, f\omega))). \tag{10}$$

We deduce that, $\psi(d(\omega, f\omega)) = 0$ or $\varphi(d(\omega, f\omega)) = 0$,that is,

$$d(\omega, f\omega) = 0,$$

this contradicts to $d(\omega, f\omega) > 0$. Therefore, we can deduce that $d(\omega, f\omega) = 0$, indicating that ω serves as a fixed point of f. Additionally, it becomes apparent that the sequence $\{f^n x_0\}$ exhibits convergence towards ω. ■

Next, we will proceed to verify the uniqueness of such a fixed point.

Theorem 3.5. *Suppose, in addition to the hypotheses of Theorem 3.2 (resp. Theorem 3.4), that*
(V): $\alpha(x, y) \geq 1$ and $\mu(x, y) \leq 1$, for all $x, y \in Fix(f)$.
Then, $Fix(f) = \{\omega\}$.

Proof. We provide a proof using contradiction. In particular, let $\omega, \upsilon \in X$ such that $\omega = f\omega$ and $\upsilon = f\upsilon$ with $\omega \neq \upsilon$. By assumption (V), we have $\alpha(\omega, \upsilon) \geq 1$ and $\mu(\omega, \upsilon) \leq 1$. Consequently, according to (2), we obtain

$$\begin{aligned}\psi(d(\omega,\upsilon)) = \psi(d(f\omega,f\upsilon)) &\leq h(\psi(d(f\omega,f\upsilon))) \\ &\leq \mathcal{F}(\mu(\omega,\upsilon),\alpha(\omega,\upsilon),\psi(M(\omega,\upsilon)),\varphi(M(\omega,\upsilon))) \\ &\leq \mathcal{F}(1,1,\psi(M(\omega,\upsilon)),\varphi(M(\omega,\upsilon))) \leq \psi(M(\omega,\upsilon)) \\ &\Longrightarrow \end{aligned}$$

$$0 < \psi(d(\omega,\upsilon)) \leq \mathcal{F}(1,1,\psi(M(\omega,\upsilon)),\varphi(M(\omega,\upsilon))) \leq \psi(M(\omega,\upsilon))$$

$$\begin{aligned} d(\omega,\upsilon) = d(f\omega,f\upsilon) &\leq M(\omega,\upsilon) \\ &= \max\left\{\begin{matrix} d(\omega,\upsilon)+|d(\omega,f\omega)-d(\upsilon,f\upsilon)|, \\ d(\omega,f\omega)+|d(\omega,\upsilon)-d(\upsilon,f\upsilon)|, \\ d(\upsilon,f\upsilon)+|d(\omega,\upsilon)-d(\omega,f\omega)| \end{matrix}\right\} = d(\omega,\upsilon), \end{aligned}$$

which leads to a contradiction. Therefore $\omega = \upsilon$. ■

The following corollary is stated by setting $\alpha(u,v)=1$, $\mu(u,v)=1$, and $h(t)=t$ in Theorem 3.4.

Corollary 3.6. *Let (X,d) be a complete metric space and $f: X \to X$ representing a mapping on X. Suppose there exists $\beta \in \mathcal{F}_s$ such that*

$$\psi(d(fx,fy)) \leq \mathcal{F}(1,1,\psi((M(x,y)),\varphi(M(x,y))) \tag{11}$$

for all $x,y \in X$, where

$$M(x,y) = \max\left\{\begin{matrix} d(x,y)+|d(x,fx)-d(y,fy)|, \\ d(x,fx)+|d(x,y)-d(y,fy)|, \\ d(y,fy)+|d(x,y)-d(x,fx)| \end{matrix}\right\}.$$

Then, $Fix(f) = \{\omega\}$ and $\{f^n x_0\}$ converges to ω for all $x_0 \in X$.

Example 3.7. Let $X = [0,\infty)$ and $d(x,y) = |x-y|$ for every $x,y \in X$, then (X,d) is a complete metric space. Let $h(l) = l$ and $\mathcal{F}(u,v,s,t) = \frac{us}{1+vt}$ for all $l,u,v,s,t \in X$. Let $\psi(t) = \frac{t}{4}$, $\varphi(t) = t^2$ then ψ,φ are an altering distance function. Let a mapping $f: X \to X$ be defined by

$$fx = \begin{cases} \dfrac{x}{6}, & \text{if } 0 \leq x \leq 1, \\ \dfrac{\sqrt{x}}{6}, & \text{if } x > 1. \end{cases}$$

Define functions $\alpha,\mu\colon X \times X \to \mathbb{R}$ as follows:

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,y \in [0,1], \\ 0, & \text{otherwise.} \end{cases} \qquad \mu(x,y) = \begin{cases} \dfrac{5}{6}, & \text{if } x,y \in [0,1], \\ 4, & \text{otherwise.} \end{cases}$$

It can be readily confirmed that f is both a triangular α-orbital admissible and a triangular μ-orbital subadmissible. Furthermore, given that f is continuous, we have $\alpha(0,f0) = \alpha(0,0) \geq 1$, and

$\mu(0, f0) = \mu(0,0) \leq 1$ for $x_0 = 0$. Our next goal is to show that f is an $\alpha - \mu - 1 - 1-$upclass contraction mapping. Referring to (2), we need to consider the following cases:
Case 1: $x, y \in [0,1]$, then

$$\begin{aligned}
M(x,y) &\leq 2,\\
1 + M^2(x,y) &\leq 5,\\
\frac{M(x,y)}{5} &\leq \frac{M(x,y)}{1+M^2(x,y)},\\
\frac{M(x,y)}{24} &\leq \frac{\frac{5}{24}M(x,y)}{1+M^2(x,y)},\\
\frac{1}{4}(\frac{|x-y|}{6}) &\leq \frac{\frac{5}{6}(\frac{M(x,y)}{4})}{1+M^2(x,y)},\\
h(\psi(d(fx,fy))) &\leq \mathcal{F}(\mu(x,y), \alpha(x,y), \psi(M(x,y)), \varphi(M(x,y))).
\end{aligned}$$

Case 2: $x, y \in (1, \infty)$, then

$$\begin{aligned}
|\sqrt{x} - \sqrt{y}| &\leq |x-y|,\\
\frac{|\sqrt{x} - \sqrt{y}|}{24} &\leq 4|x-y|,\\
\frac{1}{4}(\frac{|\sqrt{x} - \sqrt{y}|}{6}) &\leq 4M(x,y),\\
h(\psi(d(fx,fy))) &\leq \mathcal{F}(\mu(x,y), \alpha(x,y), \psi(M(x,y)), \varphi(M(x,y))).
\end{aligned}$$

Case 3: $x \in [0,1]$ and $y \in (1, \infty)$, then

$$\begin{aligned}
\sqrt{y} - x &\leq y - x,\\
\frac{|x - \sqrt{y}|}{24} &\leq 4|x-y|,\\
\frac{1}{4}(\frac{|x - \sqrt{y}|}{6}) &\leq 4M(x,y),\\
h(\psi(d(fx,fy))) &\leq \mathcal{F}(\mu(x,y), \alpha(x,y), \psi(M(x,y)), \varphi(M(x,y))).
\end{aligned}$$

This implies that f is $\alpha - \mu - 1 - 1-$upclass contraction. Since all the conditions of Theorem 3.2 are satisfied, therefore f has a fixed point, specifically $w = 0$.

References

[1] Karapinar, E., Kumam, P. & Salimi, P. (2023). On $\alpha - \psi$ -Meir-Keeler contractive mappings, *Fixed Point Theory and Applications*, 94, 1–12.

[2] Thompson, A. (1963). On certain contraction mappings in a partially ordered vector space, *Proceedings of the American Mathematical Society*, 14, 438–443.

[3] Ansari, A.H. (2014). Note on "α -admissible mappings and related fixed point theorems", *The 2nd Regional Conference on Mathematics And Applications, Payame Noor University*, 2014, 373–376.

[4] Ansari, A.H., & Kaewcharoen, A. (2016). C-class functions and fixed point theorems for generalized $\alpha-\mu-\varphi-\psi-$F-contraction type mappings in $\alpha-\mu-$complete metric spaces, *J. Nonlinear Sci. Appl.*, 9, 4177–4190.

[5] Ansari, A.H., & Shatanawi, W. (2017). C-class functions on fixed and common fixed point results for cyclic mappings of Omega-distance, *Nonlinear Anal. Model. Control*, 22, 739–753.

[6] Samet, B., Vetro C., & Vetro, P. (2012). Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Analysis, Theory, Methods and Applications*, 75, 2154–2165.

[7] Popescu, O. (2014). Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces, *Fixed Point Theory and Applications*, 190, 1–12.

[8] Ansari, A.H., Berzig M., & Chandok, S. (2015). Some fixed point theorems for (CAB)-contractive mappings and related results, *Mathematica Moravica*, 19, 97–112.

[9] Aydi, H., Felhi, A., Karapinar, E., AlRubaish, H. & Alshammari, M. (2019). Fixed points for $\alpha-\beta_E-$geraghty contractions on b-metric spaces and applications to matrix equations, *Filomat*, 33, 3737–3750.

[10] Fulga A., & Proca, A.M. (2017). Fixed points for φ_E-Geraghty contractions, *Journal of Nonlinear Science and Applications*, 10, 5125–5131.

[11] Bunpog, C., Phudolsitthiphat N., & Wiripapongsanon, A. (2023). Fixed point theorems for $\alpha-\beta_M-$Geraghty type contractions with applications in matrix equations, *Journal of Mathematical Inequalities*, 17, 1443–1461.

[12] Bakhtin, I.A. (1989). The contraction mapping principle in quasi metric spaces, *Functional Analysis*, 30, 26–37.

[13] Czerwik, S. (1998). Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Semin. Mat. Fis. Univ. Modena*, 46, 263–276.

[14] Babu G.V.R., & Sailaja, P.D. (2011). A Fixed Point Theorem of Generalized Weakly Contractive Maps in Orbitally Complete Metric Spaces, *Thai Journal of Mathematics*, 9, 1–10.

[15] Salimi, P., Latif A., & Hussain, N. (2013). Modified α-ψ-contractive mappings with applications, *Fixed Point Theory and Applications*, 2013, 1–19.

[16] Khan, M.S., Swaleh M., & Sessa, S. (1984). Fixed point theorems by altering distances between the points, *Bull. Aust. Math. Soc.*, 30, 1–9.

[17] Ansari, A.H. (2014). Note on" $\varphi-\psi-$contractive type mappings and related fixed point", *The* 2^{nd} *Regional Conference on Mathematics And Applications, PNU*, 2014, 377–380.