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Research Article

Error estimates for approximating coupled best proximity points in uniformly convex Banach spaces via the modulus of convexity

Akram Safari-Hafshejani *
Department of Pure Mathematics, Payame Noor University, Tehran, Iran

Abstract

Abstract: The study of cyclic contraction ordered pairs has garnered significant attention in recent years. We introduce the cyclic ϑ-quasi-contraction ordered pair $((\mathcal{I}, \bar{\mathcal{I}}), (\mathcal{J}, \bar{\mathcal{J}}))$ and investigate the existence and uniqueness conditions for their coupled best proximity points. Furthermore, we present a priori and a posteriori estimates of the best proximity point error of this class of cyclic contraction ordered pairs, particularly highlighting important results for cyclic ϑ-quasi-contractions in a uniformly convex Banach space with a modulus of convexity. In Paper [Zlatanov, B. (2016). Error estimates for approximating best proximity points for cyclic contractive maps. Carpathian J. Math.], the author employs geometric progressions for estimations, thus necessitating the condition that the modulus of convexity is of power type, which prevents the generalization of his results to cyclic contractions. The proofs presented in this paper enable error estimation without recourse to geometric progressions. The error calculation in an arbitrarily uniformly convex Banach space is performed solely utilizing its modulus of convexity function, and an error estimation for cyclic ϑ-contraction is provided. Consequently, this work answers the questions posed in the aforementioned paper due to B. Zlatanov.
Keywords: Coupled best proximity point, Cyclic ϑ-quasi-contraction pair, Modulus of convexity, Priori and posteriori errors estimates, UC property.
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1 Introduction and Preliminaries

In recent years, the study of cyclic contraction mappings and their ordered pair generalizations has garnered significant attention within fixed point theory [7, 12, 15, 1, 5, 6, 8, 9, 10]. These mappings have proven instrumental in extending classical results to broader settings and have provided novel insights into the existence of fixed points and best proximity points in metric spaces.

*Corresponding author: asafari@pnu.ac.ir

In this paper, we introduce the concept of cyclic ϑ-quasi-contraction for ordered pairs $((\mathcal{I},\bar{\mathcal{I}}),(\mathcal{J},\bar{\mathcal{J}}))$ and thoroughly investigate the conditions under which optimal coupled best proximity points exist and are unique. Our findings demonstrate that the well-known cyclic ϑ-quasi-contractions are indeed a specific instance within this introduced new class of ordered paired contractions. Furthermore, a substantial part of our work is dedicated to establishing a priori and a posteriori estimates of the best proximity point error for the Picard iteration method associated with this class ofcyclic contraction ordered pairs. We particularly highlight important results tailored for cyclic ϑ-quasi-contractions, demonstrating their behavior in a uniformly convex Banach space with a power modulus of convexity.

In Paper [14], the author employs geometric progressions for estimations, thus necessitating the condition that the modulus of convexity is of power type, which prevents the generalization of his results to larger classes of cyclic contractions. At the conclusion of the paper, the author poses the question of whether error estimates can be derived if the modulus of convexity is not of power type, and whether a method can be established for error estimation for cyclic ϑ-contractions. The proofs presented in this paper enable error estimation without recourse to geometric progressions. The error calculation in an arbitrarily uniformly convex Banach space is performed solely utilizing its modulus of convexity function, and an error estimation for cyclic ϑ-contraction is provided. This method provides an answer to the questions posed in the paper [14].

In the following, we present the essential definitions and concepts for the subsequent discussions. Let us begin by considering two non-empty subsets, $\mathcal{K}$ and $\mathcal{V}$ of a metric space $(\mathcal{W},\varrho)$. The term $\varrho(\mathcal{K},\mathcal{V})$ defined as $\varrho(\mathcal{K},\mathcal{V}) := \inf\{\varrho(u,v) : u \in \mathcal{K}, v \in \mathcal{V}\}$. For any pair $(u,v) \in \mathcal{K}\times\mathcal{V}$, we introduce concept $\varrho^*(u,v)$ by $\varrho^*(u,v) := \varrho(u,v) - \varrho(\mathcal{K},\mathcal{V})$. This function ϱ^* satisfies a generalized triangle inequality-like property. Specifically, for any elements $u,u' \in \mathcal{K}$ and $v \in \mathcal{V}$, the following holds:

$$\varrho^*(u,v) \leq \varrho(u,u') + \varrho^*(u',v).$$

A corresponding relation holds for elements in $\mathcal{V}$: for all $u \in \mathcal{K}$ and $v,v' \in \mathcal{V}$, we have:

$$\varrho^*(u,v) \leq \varrho^*(u,v') + \varrho(v',v).$$

Let self mapping $\mathcal{G} : \mathcal{K}\cup\mathcal{V} \to \mathcal{K}\cup\mathcal{V}$, maps $\mathcal{K}$ into $\mathcal{V}$ and $\mathcal{V}$ into $\mathcal{K}$ (i.e., $\mathcal{G}(\mathcal{K}) \subseteq \mathcal{V}$ and $\mathcal{G}(\mathcal{V}) \subseteq \mathcal{K}$), then $\mathcal{G}$ is termed a cyclic map. A cyclic map $\mathcal{G}$ satisfying the following inequality:

$$\varrho^*(\mathcal{G}u,\mathcal{G}v) \leq k\varrho^*(u,v), \tag{I}$$

for all $u \in \mathcal{K}$ and $v \in \mathcal{V}$, where $k \in [0,1)$ is a constant; is termed a cyclic contraction map [2]. An element $u^* \in \mathcal{K}$ constitutes a best proximity point of $\mathcal{G}$ if it satisfies the condition: $\varrho(u^*,\mathcal{G}u^*) = \varrho(\mathcal{K},\mathcal{V})$. Notably, if the sets $\mathcal{K}$ and $\mathcal{V}$ have a non-empty intersection, then every best proximity point of $\mathcal{G}$ is a fixed point of $\mathcal{G}$.

Definition 1.1. [3, 13] Let $\mathcal{K}$ and $\mathcal{V}$ be non-empty subsets of the metric space $(\mathcal{W},\varrho)$. The pair $(\mathcal{K},\mathcal{V})$ is said to satisfy the *UC* property whenever sequences $\{u_n\}$ and $\{u'_n\}$ in $\mathcal{K}$ and a sequence $\{v_n\}$ in $\mathcal{V}$ satisfy

$$\lim_{n\to\infty}\varrho(u_n,v_n) = \lim_{n\to\infty}\varrho(a'_n,v_n) = \varrho(\mathcal{K},\mathcal{V}),$$

it necessarily follows that:

$$\lim_{n\to\infty}\varrho(u_n,u'_n) = 0.$$

It has been established by Suzuki et al. [13] that if $\mathcal{K}$ and $\mathcal{V}$ are non-empty subsets of a uniformly convex Banach space $\mathcal{W}$ such that $\mathcal{K}$ is convex, then the pair $(\mathcal{K},\mathcal{V})$ possesses the UC property. Furthermore, the UC property holds in any metric space $(\mathcal{W},\varrho)$ if, $\varrho(\mathcal{K},\mathcal{V})=0$.

Definition 1.2. [4] The modulus of convexity of a Banach space $\mathcal{W}$ is the function $\delta_{\mathcal{W}}:[0,2]\to[0,1]$ defined by

$$\delta_{\mathcal{W}}(\epsilon)=\inf\big\{1-\|\frac{u+v}{2}\|:\ \|u\|\le 1,\ \|v\|\le 1,\ \|u-v\|\ge\epsilon\big\}.$$

The norm of the space is called uniformly convex if $\delta_{\mathcal{W}}(\epsilon)>0$ for every $\epsilon>0$. A space $(\mathcal{W},\|.\|)$ endowed with such a norm is called uniformly convex Banach space.

It is noteworthy that, within a uniformly convex Banach space $(\mathcal{W},\|\cdot\|)$, the modulus of convexity $\delta_{\mathcal{W}}$ possesses the important feature of being strictly increasing. This guarantees the existence of its inverse mapping, $\delta_{\mathcal{W}}^{-1}$, which is also strictly increasing. For the remainder of this work, a uniformly convex Banach space, along with its modulus of convexity, is represented by the triplet $(\mathcal{W},\|\cdot\|,\delta_{\mathcal{W}})$.

Remark 1.3. [4] Let $(\mathcal{W},\|\cdot\|,\delta_{\mathcal{W}})$ be a uniformly convex Banach space. For any elements $u,u',v\in\mathcal{W}$ and constants $\mathcal{R}>0$ and $r\in[0,2\mathcal{R}]$, if the conditions $\|u-v\|\le\mathcal{R}$, $\|u'-v\|\le\mathcal{R}$ and $\|u-u'\|\ge r$ are satisfied, then the following implication is valid:

$$\left\|\frac{u+u'}{2}-v\right\|\le\Big(1-\delta_{\mathcal{W}}\Big(\frac{r}{\mathcal{R}}\Big)\Big)\mathcal{R}. \tag{1}$$

Lemma 1.4. *Suppose that $\mathcal{K}$ and $\mathcal{V}$ be non-empty subsets of a uniformly convex Banach space $(\mathcal{W},\|\cdot\|,\delta_{\mathcal{W}})$ such that $\mathcal{K}$ is convex and $\varrho:=\varrho(\mathcal{K},\mathcal{V})>0$. For any elements $u,u'\in\mathcal{K}$, $v\in\mathcal{V}$ and constants $\mathcal{R}>0$ and $r\in[0,2\mathcal{R}]$, if the conditions $\|u-v\|\le\mathcal{R}$, $\|u'-v\|\le\mathcal{R}$ and $\|u-u'\|\ge r$ are satisfied, then*

$$\|u-u'\|\le\mathcal{R}\delta_{\mathcal{W}}^{-1}(\frac{\mathcal{R}-\varrho}{\mathcal{R}}).$$

Proof. Let $u,u'\in\mathcal{K}$, $v\in\mathcal{V}$, $\mathcal{R}>0$ such that

$$\begin{cases}\|u-v\|\le\mathcal{R}\\ \|u'-v\|\le\mathcal{R}.\end{cases}$$

Since $(\mathcal{W},\|\cdot\|,\delta_{\mathcal{W}})$ be a uniformly convex Banach space and $\|u-u'\|\le 2\mathcal{R}$, then

$$\varrho\le\|\frac{u+u'}{2}-v\|\le(1-\delta_{\mathcal{W}}(\frac{\|u-u'\|}{\mathcal{R}}))\mathcal{R}.$$

Also, $\delta_{\mathcal{W}}^{-1}$ is strictly increasing. So, we get

$$\|u-u'\|\le\mathcal{R}\delta_{\mathcal{W}}^{-1}(\frac{\mathcal{R}-\varrho}{\mathcal{R}}).$$

■

Definition 1.5. [15, Definition 2.1] Let $\mathcal{K}$, $\bar{\mathcal{K}}$, $\mathcal{V}$ and $\bar{\mathcal{V}}$ be non-empty subsets of the metric space $(\mathcal{W}, \varrho)$. Letting $\mathcal{I} : \mathcal{K} \times \bar{\mathcal{K}} \to \mathcal{V}$ and $\bar{\mathcal{I}} : \mathcal{K} \times \bar{\mathcal{K}} \to \bar{\mathcal{V}}$, the ordered pair $(u^*, \bar{u}^*) \in \mathcal{K} \times \bar{\mathcal{K}}$ is called a coupled best proximity point of $(\mathcal{I}, \bar{\mathcal{I}})$ if

$$\varrho^*(u^*, \mathcal{I}(u^*, \bar{u}^*)) = \varrho(\mathcal{K}, \mathcal{V}) \quad \text{and} \quad \varrho^*(\bar{u}^*, \bar{\mathcal{I}}(u^*, \bar{u}^*)) = \varrho(\bar{\mathcal{K}}, \bar{\mathcal{V}}).$$

When $\mathcal{I}(u^*, \bar{u}^*) = u^*$ and $\bar{\mathcal{I}}(u^*, \bar{u}^*) = \bar{u}^*$, then $(u^*, \bar{u}^*)$ is called a coupled fixed point of $(\mathcal{I}, \bar{\mathcal{I}})$. Letting $\mathcal{J} : \mathcal{V} \times \bar{\mathcal{V}} \to \mathcal{K}$ and $\bar{\mathcal{J}} : \mathcal{V} \times \bar{\mathcal{V}} \to \bar{\mathcal{K}}$, the ordered pair of orderer pairs $((u^*, \bar{u}^*), (v^*, \bar{v}^*)) \in (\mathcal{K} \times \bar{\mathcal{K}}) \times (\mathcal{V} \times \bar{\mathcal{V}})$ is called an optimal pair of coupled fixed point of $((\mathcal{I}, \bar{\mathcal{I}}), (\mathcal{J}, \bar{\mathcal{J}}))$ if

$$\mathcal{I}(u^*, \bar{u}^*) = u^*, \quad \bar{\mathcal{I}}(u^*, \bar{u}^*) = \bar{u}^*, \quad \mathcal{J}(v^*, \bar{v}^*) = v^*, \quad \bar{\mathcal{J}}(v^*, \bar{v}^*) = \bar{v}^*$$

and

$$\varrho(u^*, v^*) = \varrho(\mathcal{K}, \mathcal{V}) \quad \text{and} \quad \varrho(\bar{u}^*, \bar{v}^*) = \varrho(\bar{\mathcal{K}}, \bar{\mathcal{V}}).$$

2 cyclic ϑ-quasi-contraction ordered pairs of mappings

Let I denote the identity function on $\mathbb{R}$, and let

$$\Theta := \{\theta : \mathbb{R}^+ \to \mathbb{R}^+ : \theta \text{ is strictly increasing and } I - \theta \text{ is increasing}\}.$$

For any function $\vartheta \in \Theta$, the following properties are satisfied:

- $\vartheta(s) > 0$ for all $s > 0$;
- $(I - \vartheta)(s) < s$ for all $s > 0$;
- ϑ is continuous;
- The strictly increasing nature of ϑ ensures the existence of its inverse function, ϑ^{-1}. This inverse function, ϑ^{-1}, is itself strictly increasing.

Definition 2.1. Let $\mathcal{K}$, $\bar{\mathcal{K}}$, $\mathcal{V}$, and $\bar{\mathcal{V}}$ be non-empty subsets of the metric space $(\mathcal{W}, \varrho)$. Consider the mappings $\mathcal{I} : \mathcal{K} \times \bar{\mathcal{K}} \to \mathcal{V}$, $\bar{\mathcal{I}} : \mathcal{K} \times \bar{\mathcal{K}} \to \bar{\mathcal{V}}$, $\mathcal{J} : \mathcal{V} \times \bar{\mathcal{V}} \to \mathcal{K}$, and $\bar{\mathcal{J}} : \mathcal{V} \times \bar{\mathcal{V}} \to \bar{\mathcal{K}}$. The ordered pair of ordered pairs $((\mathcal{I}, \bar{\mathcal{I}}), (\mathcal{J}, \bar{\mathcal{J}}))$ is said to be a cyclic ϑ-quasi-contraction ordered pair if there exists $\vartheta \in \Theta$ such that the following inequality holds for all $(u, \bar{u}) \in \mathcal{K} \times \bar{\mathcal{K}}$ and $(v, \bar{v}) \in \mathcal{V} \times \bar{\mathcal{V}}$:

$$\begin{aligned} \varrho^*(\mathcal{I}(u, \bar{u}), \mathcal{J}(v, \bar{v})) + \varrho^*(\bar{\mathcal{I}}(u, \bar{u}), \bar{\mathcal{J}}(v, \bar{v})) \le (I - \vartheta)\big(\max\big\{&\varrho^*(u, v) + \varrho^*(\bar{u}, \bar{v}) \\ , \varrho^*(u, \mathcal{I}(u, \bar{u})) + \varrho^*(\bar{u}, \bar{\mathcal{I}}(u, \bar{u})), &\varrho^*(v, \mathcal{J}(v, \bar{v})) + \varrho^*(\bar{v}, \bar{\mathcal{J}}(v, \bar{v}))\big\}\big). \end{aligned} \tag{2}$$

Note that throughout this article we will consider sequences $\{u_n\}$ and $\{\bar{u}_n\}$ as defined below.

Definition 2.2. [15, Definition 2.6] Let $\mathcal{K}$, $\bar{\mathcal{K}}$, $\mathcal{V}$ and $\bar{\mathcal{V}}$ be non-empty subsets of the metric space $(\mathcal{W}, \varrho)$ and $\mathcal{I} : \mathcal{K} \times \bar{\mathcal{K}} \to \mathcal{V}$, $\bar{\mathcal{I}} : \mathcal{K} \times \bar{\mathcal{K}} \to \bar{\mathcal{V}}$, $\mathcal{J} : \mathcal{V} \times \bar{\mathcal{V}} \to \mathcal{K}$, and $\bar{\mathcal{J}} : \mathcal{V} \times \bar{\mathcal{V}} \to \bar{\mathcal{K}}$ be mappings. Let the sequences $\{u_n\}$ and $\{\bar{u}_n\}$ be defined with initial values $u_0 = u \in \mathcal{K}$ and $\bar{u}_0 = \bar{u} \in \bar{\mathcal{K}}$ and

$$\begin{aligned} u_{2n+1} &:= \mathcal{I}(u_{2n}, \bar{u}_{2n}), \quad u_{2n+2} := \mathcal{J}(u_{2n+1}, \bar{u}_{2n+1}), \\ \bar{u}_{2n+1} &:= \bar{\mathcal{I}}(u_{2n}, \bar{u}_{2n}), \quad \bar{u}_{2n+2} := \bar{\mathcal{J}}(u_{2n+1}, \bar{u}_{2n+1}). \end{aligned}$$

Example 2.3. Let $\mathcal{K}$, $\bar{\mathcal{K}}$, $\mathcal{V}$, and $\bar{\mathcal{V}}$ be non-empty subsets of the metric space $(\mathcal{W}, \varrho)$. Consider the mappings $\mathcal{I} : \mathcal{K} \times \bar{\mathcal{K}} \to \mathcal{V}$, $\bar{\mathcal{I}} : \mathcal{K} \times \bar{\mathcal{K}} \to \bar{\mathcal{V}}$, $\mathcal{J} : \mathcal{V} \times \bar{\mathcal{V}} \to \mathcal{K}$, and $\bar{\mathcal{J}} : \mathcal{V} \times \bar{\mathcal{V}} \to \bar{\mathcal{K}}$. If there exist non-negative numbers α, β such that $\xi := \max\{\alpha, \beta\} < 1$ and the following inequality holds:

$$\varrho^*(\mathcal{I}(u, \bar{u}), \mathcal{J}(v, \bar{v})) + \varrho^*(\bar{\mathcal{I}}(u, \bar{u}), \bar{\mathcal{J}}(v, \bar{v})) \leq \alpha\varrho^*(u, v) + \beta\varrho^*(\bar{u}, \bar{v}),$$

for all $u \in \mathcal{K}$, $v \in \mathcal{V}$, then

$$\varrho^*(\mathcal{I}(u, \bar{u}), \mathcal{J}(v, \bar{v})) + \varrho^*(\bar{\mathcal{I}}(u, \bar{u}), \bar{\mathcal{J}}(v, \bar{v})) \leq \xi(\varrho^*(u, v) + \varrho^*(\bar{u}, \bar{v})),$$

for all $u \in \mathcal{K}$, $v \in \mathcal{V}$. So, the ordered pair of orderer pairs $((\mathcal{I}, \bar{\mathcal{I}}), (\mathcal{J}, \bar{\mathcal{J}}))$ [12] is a cyclic ϑ-quasi-contraction ordered pair with $\vartheta(t) := (1 - \xi)t$ for all $t \geq 0$.

Example 2.4. Let $\mathcal{K}$ and $\mathcal{V}$ be non-empty subsets within the metric space $(\mathcal{W}, \varrho)$, and let $x \in \mathcal{W}$ be an arbitrary element. Consider a cyclic contraction $\mathcal{G} : \mathcal{K} \cup \mathcal{V} \to \mathcal{K} \cup \mathcal{V}$ (defined in [11]) satisfying the condition:

$$\varrho^*(\mathcal{G}u, \mathcal{G}v) \leq (I - \vartheta)\big(\max\{\varrho^*(u, v), \varrho^*(u, \mathcal{G}u), \varrho^*(v, \mathcal{G}v)\}\big),$$

for all $u \in \mathcal{U}$, $v \in \mathcal{V}$. Let $\bar{\mathcal{K}} = \bar{\mathcal{V}} := \{x\}$ and define mappings $\mathcal{I} : \mathcal{K} \times \bar{\mathcal{K}} \to \mathcal{V}$, $\bar{\mathcal{I}} : \mathcal{K} \times \bar{\mathcal{K}} \to \bar{\mathcal{V}}$, $\mathcal{J} : \mathcal{V} \times \bar{\mathcal{V}} \to \mathcal{K}$, and $\bar{\mathcal{J}} : \mathcal{V} \times \bar{\mathcal{V}} \to \bar{\mathcal{K}}$ as the following:

$$\mathcal{I}(u, x) = \mathcal{G}u, \quad \bar{\mathcal{I}}(u, x) = x, \quad \mathcal{J}(v, x) = \mathcal{G}v \quad \text{and} \quad \bar{\mathcal{J}}(v, x) = x,$$

for all $u \in \mathcal{K}$ and $v \in \mathcal{V}$. So, we have

$$\varrho^*(\bar{\mathcal{I}}(u, x), \bar{\mathcal{J}}(v, x)) = \varrho^*(x, \bar{\mathcal{I}}(u, x)) = \varrho^*(x, \bar{\mathcal{J}}(v, x)) = \varrho^*(x, x) = -\varrho(\bar{\mathcal{K}}, \bar{\mathcal{V}}) = 0.$$

Consequently, $((\mathcal{I}, \bar{\mathcal{I}}), (\mathcal{J}, \bar{\mathcal{J}}))$ constitutes a cyclic ϑ-contraction. This is demonstrated by the following for all $u \in \mathcal{K}$ and $v \in \mathcal{V}$:

$$\begin{aligned}
\varrho^*(\mathcal{I}(u, x), \mathcal{J}(v, x)) + \varrho^*(\bar{\mathcal{I}}(u, x), \bar{\mathcal{J}}(v, x)) =& \varrho^*(\mathcal{G}u, \mathcal{G}v) \\
&\leq (I - \vartheta)\big(\max\{\varrho^*(u, v), \varrho^*(u, \mathcal{G}u), \varrho^*(v, \mathcal{G}v)\}\big) \\
&= (I - \vartheta)\big(\max\{\varrho^*(u, v) + \varrho^*(x, x), \varrho^*(u, \mathcal{I}(u, x)) \\
&\quad + \varrho^*(x, \bar{\mathcal{I}}(u, x)), \varrho^*(v, \mathcal{J}(v, x)) + \varrho^*(x, \bar{\mathcal{J}}(v, x))\}\big).
\end{aligned}$$

The sequences $\{u_n\}$ and $\{\bar{u}_n\}$ with initial values $u_0 = u$ and $\bar{u}_0 = x$, are as follows:

$$\begin{aligned}
u_{2n+1} &:= \mathcal{I}(u_{2n}, x) = \mathcal{G}u_{2n}, & u_{2n+2} &:= \mathcal{J}(u_{2n+1}, x) = \mathcal{G}u_{2n+1}, \\
\bar{u}_{2n+1} &:= \bar{\mathcal{I}}(u_{2n}, x) = x, & \bar{u}_{2n+2} &:= \bar{\mathcal{J}}(u_{2n+1}, x) = x.
\end{aligned}$$

So, $\{u_n\} = \{\mathcal{G}^n u_0\}$ and $\{\bar{u}_n\} = \{x\}$.

Example 2.5. Let $\mathcal{K}$ and $\mathcal{V}$ be non-empty subsets of the metric space $(\mathcal{W}, \varrho)$. Consider the mappings $\mathcal{I} : \mathcal{K} \times \mathcal{K} \to \mathcal{V}$ and $\mathcal{J} : \mathcal{V} \times \mathcal{V} \to \mathcal{K}$. The ordered pair $(\mathcal{I}, \mathcal{J})$ is said to be a cyclic contraction ordered pair if there exist non-negative numbers α, β, γ such that $\xi := \alpha + \beta + \gamma < 1$ and the following inequality holds for all $(u, \bar{u}) \in \mathcal{K} \times \mathcal{K}$ and $(v, \bar{v}) \in \mathcal{V} \times \mathcal{V}$:

$$\varrho^*(\mathcal{I}(u, \bar{u}), \mathcal{J}(v, \bar{v})) \leq \alpha\varrho^*(u, v) + \beta\varrho^*(u, \mathcal{I}(u, \bar{u})) + \gamma\varrho^*(v, \mathcal{J}(v, \bar{v})). \tag{3}$$

Let $\bar{\mathcal{K}}=\mathcal{K}$ and $\bar{\mathcal{V}}=\mathcal{V}$ and define mappings $\bar{\mathcal{I}}:\mathcal{K}\times\mathcal{K}\to\mathcal{V}$ and $\bar{\mathcal{J}}:\mathcal{V}\times\mathcal{V}\to\mathcal{K}$ as the following:

$$\bar{\mathcal{I}}(u,\bar{u})=\mathcal{I}(\bar{u},u),\quad \bar{\mathcal{J}}(v,\bar{v})=\mathcal{J}(\bar{v},v),$$

for all $(u,\bar{u})\in\mathcal{K}\times\mathcal{K}$ and $(v,\bar{v})\in\mathcal{V}\times\mathcal{V}$. From (3), we have

$$\begin{aligned}\varrho^*(\bar{\mathcal{I}}(u,\bar{u}),\bar{\mathcal{J}}(v,\bar{v}))&=\varrho^*(\mathcal{I}(\bar{u},u),\mathcal{J}(\bar{v},v))\\&\le\alpha\varrho^*(\bar{u},\bar{v})+\beta\varrho^*(\bar{u},\bar{\mathcal{I}}(u,\bar{u}))+\gamma\varrho^*(\bar{v},\bar{\mathcal{J}}(v,\bar{v})).\end{aligned}$$

Hence

$$\begin{aligned}\varrho^*(\mathcal{I}(u,\bar{u}),\mathcal{J}(v,\bar{v}))+\varrho^*(\bar{\mathcal{I}}(u,\bar{u}),\bar{\mathcal{J}}(v,\bar{v}))&\le\alpha(\varrho^*(u,v)+\varrho^*(\bar{u},\bar{v}))\\&+\beta(\varrho^*(u,\mathcal{I}(u,\bar{u}))+\varrho^*(\bar{u},\bar{\mathcal{I}}(u,\bar{u})))+\gamma(\varrho^*(v,\mathcal{J}(v,\bar{v}))+\varrho^*(\bar{v},\bar{\mathcal{J}}(v,\bar{v})))\\&\le\xi\max\big\{\varrho^*(u,v)+\varrho^*(\bar{u},\bar{v}),\varrho^*(u,\mathcal{I}(u,\bar{u}))+\varrho^*(\bar{u},\bar{\mathcal{I}}(u,\bar{u}))\\&\qquad,\varrho^*(v,\mathcal{J}(v,\bar{v}))+\varrho^*(\bar{v},\bar{\mathcal{J}}(v,\bar{v}))\big\}.\end{aligned}$$

So, the ordered pair of orderer pairs $((\mathcal{I},\bar{\mathcal{I}}),(\mathcal{J},\bar{\mathcal{J}}))$ is a cyclic ϑ-quasi-contraction ordered pair with $\vartheta(t):=(1-\xi)t$ for all $t\ge 0$.

We continue this section with the following fundamental and applicable lemma.

Lemma 2.6. *Assume $\mathcal{K}$, $\bar{\mathcal{K}}$, $\mathcal{V}$, and $\bar{\mathcal{V}}$ are non-empty subsets of the metric space $(\mathcal{W},\varrho)$. Let $\mathcal{I}:\mathcal{K}\times\bar{\mathcal{K}}\to\mathcal{V}$, $\bar{\mathcal{I}}:\mathcal{K}\times\bar{\mathcal{K}}\to\bar{\mathcal{V}}$, $\mathcal{J}:\mathcal{V}\times\bar{\mathcal{V}}\to\mathcal{K}$, and $\bar{\mathcal{J}}:\mathcal{V}\times\bar{\mathcal{V}}\to\bar{\mathcal{K}}$ be mappings such that the combination $((\mathcal{I},\bar{\mathcal{I}}),(\mathcal{J},\bar{\mathcal{J}}))$ be a cyclic ϑ-quasi-contraction ordered pair. Then, the following assertions hold:*

(i) *For any pair of natural numbers m and n satisfying the condition $n\ge m$, we can state that:*

$$0\le\varrho^*(u_{2m},u_{2n+1})+\varrho^*(\bar{u}_{2m},\bar{u}_{2n+1})\le(I-\vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0}),$$

where $\mathcal{R}_{u_0,\bar{u}_0}:=\vartheta^{-1}(\varrho(u_0,u_2)+\varrho(\bar{u}_0,\bar{u}_2))$.

(ii) *For every positive real number ϵ, there must exist an index $m\in\mathbb{N}$ such that for all n with $n\ge m$, we have: $\varrho(u_{2m},u_{2n+1})\le\varrho(\mathcal{K},\mathcal{V})+\epsilon$ and $\varrho(\bar{u}_{2m},\bar{u}_{2n+1})\le\varrho(\bar{\mathcal{K}},\bar{\mathcal{V}})+\epsilon$.*

Proof. Let $n\in\mathbb{N}$ and

$$\delta[\Omega^*(u_0,\bar{u}_0,n)]:=\max\{\varrho^*(u_{2i},u_{2j+1})+\varrho^*(\bar{u}_{2i},\bar{u}_{2j+1})\ :\quad 0\le i,j\le n\}.$$

Our primary step is to establish that for every natural number n, the following relationship holds:

$$\delta[\Omega^*(u_0,\bar{u}_0,n)]=\varrho^*(u_0,u_{2j+1})+\varrho^*(\bar{u}_0,\bar{u}_{2j+1}),\ \text{where}\ 0\le j\le n. \tag{4}$$

Let us assume that $\delta[\Omega^*(u_0,\bar{u}_0,n)]=\varrho^*(u_{2i},u_{2j+1})+\varrho^*(\bar{u}_{2i},\bar{u}_{2j+1})$, with the constraints $1\le i\le n$ and $0\le j\le n$. Given that $((\mathcal{I},\bar{\mathcal{I}}),(\mathcal{J},\bar{\mathcal{J}}))$ is a cyclic ϑ-contraction pair, as defined by relation (2), we can write the following sequence of inequalities:

$$\begin{aligned}\varrho^*(u_{2i},u_{2j+1})+\varrho^*(\bar{u}_{2i},\bar{u}_{2j+1}) &= \varrho^*(\mathcal{I}(u_{2j},\bar{u}_{2j}),\mathcal{J}(u_{2i-1},\bar{u}_{2i-1}))+\varrho^*(\bar{\mathcal{I}}(u_{2j},\bar{u}_{2j}),\bar{\mathcal{J}}(u_{2i-1},\bar{u}_{2i-1}))\\ &\leq (I-\vartheta)\max\Big\{\varrho^*((u_{2i-1},u_{2j})+\varrho^*(\bar{u}_{2i-1},\bar{u}_{2j}))\\ &\quad ,\varrho^*(u_{2j},\mathcal{I}(u_{2j},\bar{u}_{2j}))+\varrho^*(\bar{u}_{2j},\bar{\mathcal{I}}(u_{2j},\bar{u}_{2j}))\\ &\quad ,\varrho^*(u_{2i-1},\mathcal{J}(u_{2i-1},\bar{u}_{2i-1}))+\varrho^*(\bar{u}_{2i-1},\bar{\mathcal{J}}(u_{2i-1},\bar{u}_{2i-1}))\Big\}\\ &\leq (I-\vartheta)\max\Big\{\varrho^*((u_{2i-1},u_{2j})+\varrho^*(\bar{u}_{2i-1},\bar{u}_{2j}))\\ &\quad ,\varrho^*(u_{2j},u_{2j+1})+\varrho^*(\bar{u}_{2j},\bar{u}_{2j+1}),\varrho^*(u_{2i-1},u_{2i})+\varrho^*(\bar{u}_{2i-1},\bar{u}_{2i})\Big\}\\ &\leq (I-\vartheta)(\delta[\Omega^*(u_0,\bar{u}_0,n)]).\end{aligned}$$

The preceding inequalities lead directly to the conclusion that $\vartheta(\delta[\Omega^*(u_0,\bar{u}_0,n)])\leq 0$. Since ϑ is a strictly increasing function, this inequality forces $\delta[\Omega^*(u_0,\bar{u}_0,n)]=0$. Hence, we must have $\delta[\Omega^*(u_0,\bar{u}_0,n)]=\varrho^*(u_0,u_1)+\varrho^*(\bar{u}_0,\bar{u}_1)$. Consequently, this confirms the validity of equation (4).

We shall now demonstrate that for all natural numbers n, the following inequality holds true:

$$\delta[\Omega^*(u_0,\bar{u}_0,n)]\leq \mathcal{R}_{u_0,\bar{u}_0}. \tag{5}$$

Consider the case where $\delta[\Omega^*(u_0,\bar{u}_0,n)]=\varrho^*(u_0,u_1)+\varrho^*(\bar{u}_0,\bar{u}_1)$. By applying the triangular inequality, we establish the subsequent chain of relations:

$$\begin{aligned}\delta[\Omega^*(u_0,\bar{u}_0,n)] &= \varrho^*(u_0,u_1)+\varrho^*(\bar{u}_0,\bar{u}_1)\\ &\leq \varrho(u_0,u_2)+\varrho(\bar{u}_0,\bar{u}_2)+\varrho^*(u_2,u_1)+\varrho^*(\bar{u}_2,\bar{u}_1)\\ &\leq \varrho(u_0,u_2)+\varrho(\bar{u}_0,\bar{u}_2)+(I-\vartheta)(\delta[\Omega^*(u_0,\bar{u}_0,n)]).\end{aligned}$$

Under these conditions, the result we obtain is:

$$\delta[\Omega^*(u_0,\bar{u}_0,n)]\leq \vartheta^{-1}(\varrho(u_0,u_2)+\varrho(\bar{u}_0,\bar{u}_2)).$$

Next, we analyze the situation where $\delta[\Omega^*(u_0,\bar{u}_0,n)]=\varrho^*(u_0,u_{2j+1})+\varrho^*(\bar{u}_0,\bar{u}_{2j+1}))$, with the constraint $1<j\leq n$. A similar application of the triangle inequality yields:

$$\begin{aligned}\delta[\Omega^*(u_0,\bar{u}_0,n)] &= \varrho^*(u_0,u_{2j+1})+\varrho^*(\bar{u}_0,\bar{u}_{2j+1}))\\ &\leq \varrho(u_0,u_2)+\varrho(\bar{u}_0,\bar{u}_2)+\varrho^*(u_2,v_{2j+1})+\varrho^*(\bar{u}_2,\bar{v}_{2j+1})\\ &\leq \varrho(u_0,u_2)+\varrho(\bar{u}_0,\bar{u}_2)+(I-\vartheta)(\delta[\Omega^*(u_0,\bar{u}_0,n)]).\end{aligned}$$

In this particular scenario, the conclusion remains the same:

$$\delta[\Omega^*(u_0,\bar{u}_0,n)]\leq \vartheta^{-1}(\varrho(u_0,u_2)+\varrho(\bar{u}_0,\bar{u}_2)).$$

So, the fulfillment of inequality (5) is confirmed. To proceed with the proof of part (i), we note that for some $m \leq i, j \leq m+n$, we have

$$\begin{aligned}\delta[\Omega^*(u_{2m}, \bar{u}_{2m}, 2n+1)] &= \varrho^*(u_{2i}, u_{2j+1}) + \varrho^*(\bar{u}_{2i}, \bar{u}_{2j+1}) \\ &\leq (I-\vartheta)\max\Big\{\varrho^*((u_{2i-1}, u_{2j}) + \varrho^*(\bar{u}_{2i-1}, \bar{u}_{2j})) \\ &\qquad , \varrho^*(u_{2j}, u_{2j+1}) + \varrho^*(\bar{u}_{2j}, \bar{u}_{2j+1}), \varrho^*(u_{2i-1}, u_{2i}) + \varrho^*(\bar{u}_{2i-1}, \bar{u}_{2i})\Big\} \\ &\leq (I-\vartheta)^2(\delta[\Omega^*(u_{2m-2}, \bar{u}_{2m-2}, 2n+3)]).\end{aligned}$$

Drawing upon the preceding results, for all $n \geq m \geq 0$, we deduce that:

$$\begin{aligned}0 \leq \varrho^*(u_{2m}, u_{2n+1}) + \varrho^*(\bar{u}_{2m}, \bar{u}_{2n+1}) &\leq (I-\vartheta)^2(\delta[\Omega^*(u_{2m-2}, \bar{u}_{2m-2}, 2n+3)]) \\ &\leq (I-\vartheta)^4(\delta[\Omega^*(u_{2m-4}, \bar{u}_{2m-4}, 2n+5)]) \\ &\vdots \\ &\leq (I-\vartheta)^{2m}(\delta[\Omega^*(u_0, \bar{u}_0, 2n+2m+1)]) \\ &\leq (I-\vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0}).\end{aligned} \tag{6}$$

(ii) Turning to the second part, based on the inherent characteristics of ϑ and relation (6), we observe that for any $n \in \mathbb{N}$:

$$\mathcal{R}_{u_0,\bar{u}_0} \geq (I-\vartheta)(\mathcal{R}_{u_0,\bar{u}_0}) \geq (I-\vartheta)^2(\mathcal{R}_{u_0,\bar{u}_0}) \geq \cdots \geq 0.$$

This demonstrates that the sequence of terms given by $\{(I-\vartheta)^k(\mathcal{R}_{u_0,\bar{u}_0})\}$ is non-increasing and bounded. Let us $\lim_{k\to\infty}(I-\vartheta)^k(\mathcal{R}_{u_0,\bar{u}_0}) = s$, where $s \geq 0$. From the continuity of the operator $I-\vartheta$, we find that:

$$(I-\vartheta)(s) = s.$$

This equality simplifies to $\vartheta(s) = 0$. Utilizing the known properties of ϑ, specifically its strictly increasing nature, we conclude that $s = 0$. The completion of the proof is now achieved by referencing the result established in part (i). ■

The next lemma is a direct consequence of Lemma 2.6(ii).

Lemma 2.7. *Within the metric space $(\mathcal{W}, \varrho)$, assume that $\mathcal{K}$, $\bar{\mathcal{K}}$, $\mathcal{V}$, and $\bar{\mathcal{V}}$ are non-empty subsets. The mappings $\mathcal{I} : \mathcal{K}\times\bar{\mathcal{K}} \to \mathcal{V}$, $\bar{\mathcal{I}} : \mathcal{K}\times\bar{\mathcal{K}} \to \bar{\mathcal{V}}$, $\mathcal{J} : \mathcal{V}\times\bar{\mathcal{V}} \to \mathcal{K}$, and $\bar{\mathcal{J}} : \mathcal{V}\times\bar{\mathcal{V}} \to \bar{\mathcal{K}}$ are defined such that the composition $((\mathcal{I}, \bar{\mathcal{I}}), (\mathcal{J}, \bar{\mathcal{J}}))$ is a cyclic ϑ-quasi-contraction ordered pair. Then, if the UC property applies to any of the pairs $(\mathcal{K}, \mathcal{V})$, $(\mathcal{V}, \mathcal{K})$, $(\bar{\mathcal{K}}, \bar{\mathcal{V}})$, or $(\bar{\mathcal{V}}, \bar{\mathcal{K}})$, then the associated sequence namely $\{u_{2n}\}$, $\{u_{2n+1}\}$, $\{\bar{u}_{2n}\}$, or $\{\bar{u}_{2n+1}\}$ in order is a Cauchy sequence.*

We now have the necessary tools to prove our first main result. The subsequent theorem focuses on establishing the existence and uniqueness of a coupled best proximity point pertaining to a cyclic ϑ-quasi-contraction ordered pair $((\mathcal{I}, \bar{\mathcal{I}}), (\mathcal{J}, \bar{\mathcal{J}}))$.

Theorem 2.8. *Let $\mathcal{K}$, $\bar{\mathcal{K}}$, $\mathcal{V}$ and $\bar{\mathcal{V}}$ be non-empty closed subsets of the complete metric space $(\mathcal{W}, \varrho)$ such that the pairs $(\mathcal{K}, \mathcal{V})$ and $(\bar{\mathcal{K}}, \bar{\mathcal{V}})$ have the UC property. Let $\mathcal{I} : \mathcal{K} \times \bar{\mathcal{K}} \to \mathcal{V}$, $\bar{\mathcal{I}} : \mathcal{K} \times \bar{\mathcal{K}} \to \bar{\mathcal{V}}$, $\mathcal{J} : \mathcal{V} \times \bar{\mathcal{V}} \to \mathcal{K}$ and $\bar{\mathcal{J}} : \mathcal{V} \times \bar{\mathcal{V}} \to \bar{\mathcal{K}}$ be mappings such that $((\mathcal{I}, \bar{\mathcal{I}}), (\mathcal{J}, \bar{\mathcal{J}}))$ be a cyclic ϑ-quasi-contraction ordered pair. Then $(\mathcal{I}, \bar{\mathcal{I}})$ has a unique coupled best proximity point $(u^*, \bar{u}^*) \in \mathcal{K} \times \bar{\mathcal{K}}$ such that for all $(u_0, \bar{u}_0) \in \mathcal{K} \times \bar{\mathcal{K}}$ there hold*

$$u_{2n} \to u^* \quad \text{and} \quad \bar{u}_{2n} \to \bar{u}^*.$$

Proof. Lemma 2.7 guarantees the existence of $u^* \in \mathcal{K}$ and $\bar{u}^* \in \bar{\mathcal{K}}$ such that

$$u_{2n} \to u^* \quad \text{and} \quad \bar{u}_{2n} \to \bar{u}^*.$$

Now, from Lemma 2.6(ii), we have

$$\begin{aligned}
\varrho^*(\mathcal{I}(u^*, \bar{u}^*), u^*) + \varrho^*(\bar{\mathcal{I}}(u^*, \bar{u}^*), \bar{u}^*) &= \lim_{n\to\infty} \big(\varrho^*(\mathcal{I}(u^*, \bar{u}^*), u_{2n}) + \varrho^*(\bar{\mathcal{I}}(u^*, \bar{u}^*), \bar{u}_{2n})\big) \\
&= \lim_{n\to\infty} \big(\varrho^*(\mathcal{I}(u^*, \bar{u}^*), \mathcal{J}(u_{2n-1}, \bar{u}_{2n-1})) + \varrho^*(\bar{\mathcal{I}}(u^*, \bar{u}^*), \bar{\mathcal{J}}(u_{2n-1}, \bar{u}_{2n-1}))\big) \\
&\leq \lim_{n\to\infty} (I - \vartheta)\big(\max\big\{\varrho^*(u^*, u_{2n-1}) + \varrho^*(\bar{u}^*, \bar{u}_{2n-1}), \varrho^*(u^*, \mathcal{I}(u^*, \bar{u}^*)) + \varrho^*(\bar{u}^*, \bar{\mathcal{I}}(u^*, \bar{u}^*)) \\
&\qquad , \varrho^*(u_{2n-1}, u_{2n}) + \varrho^*(\bar{u}_{2n-1}, \bar{u}_{2n})\big\}\big) \\
&= (I - \vartheta)\big(\varrho^*(u^*, \mathcal{I}(u^*, \bar{u}^*)) + \varrho^*(\bar{u}^*, \bar{\mathcal{I}}(u^*, \bar{u}^*))\big),
\end{aligned}$$

hence, $\vartheta\big(\varrho^*(u^*, \mathcal{I}(u^*, \bar{u}^*)) + \varrho^*(\bar{u}^*, \bar{\mathcal{I}}(u^*, \bar{u}^*))\big) = 0$. So we get $\varrho(u^*, \mathcal{I}(u^*, \bar{u}^*)) = \varrho(\mathcal{K}, \mathcal{V})$ and $\varrho^*(\bar{\mathcal{I}}(u^*, \bar{u}^*), \bar{u}^*) = \varrho(\mathcal{K}, \mathcal{V})$.

For uniqueness, suppose that $(z^*, \bar{z}^*) \in \mathcal{K} \times \bar{\mathcal{K}}$ is another coupled best proximity point of $(\mathcal{I}, \bar{\mathcal{I}})$. Using (2), it is easy to prove that

$$\varrho^*\Big(\mathcal{J}\big(\mathcal{I}(u^*, \bar{u}^*), \bar{\mathcal{I}}(u^*, \bar{u}^*)\big), \mathcal{I}(u^*, \bar{u}^*)\Big) + \varrho^*\Big(\bar{\mathcal{J}}\big(\mathcal{I}(u^*, \bar{u}^*), \bar{\mathcal{I}}(u^*, \bar{u}^*)\big), \bar{\mathcal{I}}(u^*, \bar{u}^*)\Big) = 0, \tag{7}$$

Since $(\mathcal{K}, \mathcal{V})$ and $(\bar{\mathcal{K}}, \bar{\mathcal{V}})$ possess the UC property, it follows from (7) that $\mathcal{J}\big(\mathcal{I}(u^*, \bar{u}^*), \bar{\mathcal{I}}(u^*, \bar{u}^*)\big) = u^*$ and $\bar{\mathcal{J}}\big(\mathcal{I}(u^*, \bar{u}^*), \bar{\mathcal{I}}(u^*, \bar{u}^*)\big) = \bar{u}^*$. A similar argument establishes that $\mathcal{J}\big(\mathcal{I}(z^*, \bar{z}^*), \bar{\mathcal{I}}(z^*, \bar{z}^*)\big) = z^*$ and $\bar{\mathcal{J}}\big(\mathcal{I}(z^*, \bar{z}^*), \bar{\mathcal{I}}(z^*, \bar{z}^*)\big) = \bar{z}^*$. Consequently, by using (2), we have

$$\begin{aligned}
&\varrho^*(u^*, \mathcal{I}(z^*, \bar{z}^*)) + \varrho^*(\bar{u}^*, \bar{\mathcal{I}}(z^*, \bar{z}^*)) \\
&\quad = \varrho^*\Big(\mathcal{J}\big(\mathcal{I}(u^*, \bar{u}^*), \bar{\mathcal{I}}(u^*, \bar{u}^*)\big), \mathcal{I}(z^*, \bar{z}^*)\Big) + \varrho^*\Big(\bar{\mathcal{J}}\big(\mathcal{I}(u^*, \bar{u}^*), \bar{\mathcal{I}}(u^*, \bar{u}^*)\big), \bar{\mathcal{I}}(z^*, \bar{z}^*)\Big) \\
&\quad \leq (I - \vartheta)\Big(\varrho^*(z^*, \mathcal{I}(u^*, \bar{u}^*)) + \varrho^*(\bar{z}^*, \bar{\mathcal{I}}(u^*, \bar{u}^*))\Big) \\
&\quad = (I - \vartheta)\Big(\varrho^*\Big(\mathcal{J}\big(\mathcal{I}(z^*, \bar{z}^*), \bar{\mathcal{I}}(z^*, \bar{z}^*)\big), \mathcal{I}(u^*, \bar{u}^*)\Big) + \varrho^*\Big(\bar{\mathcal{J}}\big(\mathcal{I}(z^*, \bar{z}^*), \bar{\mathcal{I}}(z^*, \bar{z}^*)\big), \bar{\mathcal{I}}(u^*, \bar{u}^*)\Big)\Big) \\
&\quad \leq (I - \vartheta)\Big(\varrho^*(u^*, \mathcal{I}(z^*, \bar{z}^*)) + \varrho^*(\bar{u}^*, \bar{\mathcal{I}}(z^*, \bar{z}^*))\Big),
\end{aligned}$$

Therefore, we obtain $\varrho(u^*, \mathcal{I}(z^*, \bar{z}^*)) = \varrho(z^*, \mathcal{I}(z^*, \bar{z}^*)) = \varrho(\mathcal{K}, \mathcal{V})$. The UC property of $(\mathcal{K}, \mathcal{V})$ implies that $z^* = u^*$. Similarly, we can prove that $\bar{z}^* = \bar{u}^*$. ∎

3 On the error estimates of successive approximation sequences

Consider sequences $\{u_n\}$ and $\{\bar{u}_n\}$ as the same sequence introduced in the previous section. To achieve our goals in this section, for all $n \geq 0$ consider $\mathcal{R}_{u_0,\bar{u}_0}$ and $\mathcal{R}_{u_n,\bar{u}_n}$ as follow:

$$\mathcal{R}_{u_0,\bar{u}_0} := \vartheta^{-1}(\varrho(u_0, u_2) + \varrho(\bar{u}_0, \bar{u}_2))$$

and

$$\mathcal{R}_{u_n,\bar{u}_n} := \vartheta^{-1}(\varrho(u_n, u_{n+2}) + \varrho(\bar{u}_n, \bar{u}_{n+2})).$$

Also, assume that $\varrho := \varrho(\mathcal{K}, \mathcal{V})$ and $\bar{\varrho} := \varrho(\bar{\mathcal{K}}, \bar{\mathcal{V}})$.

Theorem 3.1. *Let $\mathcal{K}$, $\bar{\mathcal{K}}$, $\mathcal{V}$ and $\bar{\mathcal{V}}$ be non-empty closed subsets of the complete metric space $(\mathcal{W}, \varrho)$ and $\mathcal{I} : \mathcal{K} \times \bar{\mathcal{K}} \to \mathcal{V}$, $\bar{\mathcal{I}} : \mathcal{K} \times \bar{\mathcal{K}} \to \bar{\mathcal{V}}$, $\mathcal{J} : \mathcal{V} \times \bar{\mathcal{V}} \to \mathcal{K}$ and $\bar{\mathcal{J}} : \mathcal{V} \times \bar{\mathcal{V}} \to \bar{\mathcal{K}}$ be mappings. Let there exists $\vartheta \in \Theta$ such that there holds the inequality*

$$\begin{aligned}\varrho(\mathcal{I}(u,\bar{u}), \mathcal{J}(v,\bar{v})) + \varrho(\bar{\mathcal{I}}(u,\bar{u}), \bar{\mathcal{J}}(v,\bar{v})) \leq (I - \vartheta)\big(&\max\big\{\varrho(u,v) + \varrho(\bar{u},\bar{v}), \varrho(u, \mathcal{I}(u,\bar{u})) \\ &+ \varrho(\bar{u}, \bar{\mathcal{I}}(u,\bar{u})), \varrho(v, \mathcal{J}(v,\bar{v})) + \varrho(\bar{v}, \bar{\mathcal{J}}(v,\bar{v}))\big\}\big).\end{aligned}$$

for all $(u, \bar{u}) \in \mathcal{K} \times \bar{\mathcal{K}}$ and $(v, \bar{v}) \in \mathcal{V} \times \bar{\mathcal{V}}$. Then

(i) $((\mathcal{I}, \bar{\mathcal{I}}), (\mathcal{J}, \bar{\mathcal{J}}))$ has a unique common coupled fixed point $(u^*, \bar{u}^*) \in (\mathcal{K} \times \bar{\mathcal{K}}) \cap (\mathcal{V} \times \bar{\mathcal{V}})$ such that for all $(u_0, \bar{u}_0) \in (\mathcal{K} \times \bar{\mathcal{K}}) \cup (\mathcal{V} \times \bar{\mathcal{V}})$ there hold

$$u_n \to u^* \quad \text{and} \quad \bar{u}_n \to \bar{u}^*;$$

(ii) a priori error estimate holds in the following implication:

$$\varrho(u^*, u_n) + \varrho(\bar{u}^*, \bar{u}_n) \leq (I - \vartheta)^n(\mathcal{R}_{u_0,\bar{u}_0});$$

(iii) a posteriori error estimate holds in the following implication:

$$\varrho(u^*, u_n) + \varrho(\bar{u}^*, \bar{u}_n) \leq \mathcal{R}_{u_n,\bar{u}_n}.$$

Proof. Exactly the same as the proof of Lemma 2.6, for all natural numbers m and n with $n \geq m \geq 0$, we can prove that

$$0 \leq \varrho^*(u_{2m}, u_{2n+1}) + \varrho^*(\bar{u}_{2m}, \bar{u}_{2n+1}) \leq (I - \vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0}), \tag{8}$$

this implies that $\varrho = \bar{\varrho} = 0$. Now, the assertions are derived from (8) and Theorem 2.8. ■

Let $\mathcal{K}$, $\bar{\mathcal{K}}$, $\mathcal{V}$, and $\bar{\mathcal{V}}$ be non-empty, closed, and convex subsets of a uniformly convex Banach space $(\mathcal{W}, \|\cdot\|, \delta_{\mathcal{W}})$. Then all conditions of Theorem 2.8 are then satisfied, yielding the existence and uniqueness of an optimal pair of coupled fixed points for $((\mathcal{I}, \bar{\mathcal{I}}), (\mathcal{J}, \bar{\mathcal{J}}))$. The next theorem, like in [12, 15, 14], aims to establish a priori and a posteriori error estimates for the Picard iteration $(\mathcal{I}, \bar{\mathcal{I}})^n(u_0, \bar{u}_0) = (u_n, \bar{u}_n)$ when, $\varrho := \varrho(\mathcal{K}, \mathcal{V}) > 0$ and $\bar{\varrho} := \varrho(\bar{\mathcal{K}}, \bar{\mathcal{V}}) > 0$.

Theorem 3.2. *Let $\mathcal{K}$, $\bar{\mathcal{K}}$, $\mathcal{V}$ and $\bar{\mathcal{V}}$ be non-empty closed and convex subsets of a uniformly convex Banach space $(\mathcal{W}, \|\cdot\|, \delta_{\mathcal{W}})$. Let $\mathcal{I}: \mathcal{K}\times\bar{\mathcal{K}} \to \mathcal{V}$, $\bar{\mathcal{I}}: \mathcal{K}\times\bar{\mathcal{K}} \to \bar{\mathcal{V}}$, $\mathcal{J}: \mathcal{V}\times\bar{\mathcal{V}} \to \mathcal{K}$ and $\bar{\mathcal{J}}: \mathcal{V}\times\bar{\mathcal{V}} \to \bar{\mathcal{K}}$ be mappings such that $((\mathcal{I},\bar{\mathcal{I}}),(\mathcal{J},\bar{\mathcal{J}}))$ be a cyclic ϑ-quasi-contraction ordered pair. Then*

(i) *$(\mathcal{I},\bar{\mathcal{I}})$ has a unique coupled best proximity point $(u^*,\bar{u}^*) \in \mathcal{K}\times\bar{\mathcal{K}}$ such that for all $(u_0,\bar{u}_0) \in \mathcal{K}\times\bar{\mathcal{K}}$ there hold*

$$u_{2n} \to u^* \quad \text{and} \quad \bar{u}_{2n} \to \bar{u}^*.$$

(ii) *when $\varrho = \varrho(\mathcal{K},\mathcal{V}) > 0$, a priori error estimate and a posteriori error estimate hold in the following implications, respectively:*

(a) $\|u^* - u_{2n}\| \leq (\mathcal{R}_{u_0,\bar{u}_0} + \varrho)\, \delta_{\mathcal{W}}^{-1}\left(\frac{(I-\vartheta)^{2n}(\mathcal{R}_{u_0,\bar{u}_0})}{\varrho}\right)$;

(b) $\|u_{2n} - u^*\| \leq (\mathcal{R}_{u_{2n},\bar{u}_{2n}} + \varrho)\, \delta_{\mathcal{W}}^{-1}\left(\frac{\mathcal{R}_{u_{2n},\bar{u}_{2n}}}{\mathcal{R}_{u_{2n},\bar{u}_{2n}}+\varrho}\right)$;

(iii) *when $\bar{\varrho} = \varrho(\bar{\mathcal{K}},\bar{\mathcal{V}}) > 0$, a priori error estimate and a posteriori error estimate hold in the following implications, respectively:*

(a) $\|\bar{u}^* - \bar{u}_{2n}\| \leq (\mathcal{R}_{u_0,\bar{u}_0} + \bar{\varrho})\, \delta_{\mathcal{W}}^{-1}\left(\frac{(I-\vartheta)^{2n}(\mathcal{R}_{u_0,\bar{u}_0})}{\bar{\varrho}}\right)$;

(b) $\|\bar{u}_{2n} - \bar{u}^*\| \leq (\mathcal{R}_{u_{2n},\bar{u}_{2n}} + \bar{\varrho})\, \delta_{\mathcal{W}}^{-1}\left(\frac{\mathcal{R}_{u_{2n},\bar{u}_{2n}}}{\mathcal{R}_{u_{2n},\bar{u}_{2n}}+\bar{\varrho}}\right)$;

Proof. The proof of (i) follows directly from Theorem 2.8. (ii)(a) From Lemma 2.6, for $n \geq m \geq 0$ we have the inequalities

$$\|u_{2m} - u_{2n+1}\| \leq (I-\vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0}) + \varrho$$

and

$$\|u_{2n} - u_{2n+1}\| \leq (I-\vartheta)^{2n}(\mathcal{R}_{u_0,\bar{u}_0}) + \varrho \leq (I-\vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0}) + \varrho.$$

Now from Lemma 1.4 with $u = u_{2n}$, $u' = u_{2m}$, $v = u_{2n+1}$ and $\mathcal{R} = (I-\vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0}) + \varrho$, we can derive the chain of inequalities

$$\begin{aligned}\|u_{2n} - u_{2m}\| &\leq \left((I-\vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0}) + \varrho\right) \delta_{\mathcal{W}}^{-1}\left(\frac{(I-\vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0})}{(I-\vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0}) + \varrho}\right)\\ &\leq (\mathcal{R}_{u_0,\bar{u}_0} + \varrho)\, \delta_{\mathcal{W}}^{-1}\left(\frac{(I-\vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0})}{\varrho}\right).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\|u^* - u_{2m}\| \leq (\mathcal{R}_{u_0,\bar{u}_0} + \varrho)\, \delta_{\mathcal{W}}^{-1}\left(\frac{(I-\vartheta)^{2m}(\mathcal{R}_{u_0,\bar{u}_0})}{\varrho}\right).$$

(ii)(b) In a similar way, we have

$$\|u_{2n} - u_{2m}\| \leq (\mathcal{R}_{u_{2m},\bar{u}_{2m}} + \varrho)\, \delta_{\mathcal{W}}^{-1} \left(\frac{(I-\vartheta)^0(\mathcal{R}_{u_{2m},\bar{u}_{2m}})}{(I-\vartheta)^0(\mathcal{R}_{u_{2m},\bar{u}_{2m}}) + \varrho} \right).$$

Letting $n \to \infty$, we obtain

$$\|u^* - u_{2m}\| \leq (\mathcal{R}_{u_{2m},\bar{u}_{2m}} + \varrho)\, \delta_{\mathcal{W}}^{-1} \left(\frac{\mathcal{R}_{u_{2m},\bar{u}_{2m}}}{\mathcal{R}_{u_{2m},\bar{u}_{2m}} + \varrho} \right).$$

The proof of other results is similar. ∎

Combining the results from Example 2.4 and Theorem 3.2, we establish Theorem 4.5 in [11], a key finding of that work.

Corollary 3.3. Let $\mathcal{K}$ and $\mathcal{V}$ be non-empty closed and convex subsets of a uniformly convex Banach space $(\mathcal{W}, \|\cdot\|, \delta_{\mathcal{W}})$ such that $\varrho = \varrho(\mathcal{K}, \mathcal{V}) > 0$. Consider a cyclic contraction $\mathcal{G} : \mathcal{K} \cup \mathcal{V} \to \mathcal{K} \cup \mathcal{V}$ satisfying the condition:

$$\varrho^*(\mathcal{G}u, \mathcal{G}v) \leq (I-\vartheta)\big(\max\{\varrho^*(u,v), \varrho^*(u,\mathcal{G}u), \varrho^*(v,\mathcal{G}v)\}\big),$$

for all $u \in \mathcal{K}$, $v \in \mathcal{V}$ and some $\vartheta \in \Theta$. Then

(i) there exists a unique best proximity point u^* of $\mathcal{G}$ in $\mathcal{K}$ such that there holds $\mathcal{G}^{2n}u_0 \to u^*$ for every $u_0 \in \mathcal{K}$.

(ii) a priori error estimate holds in the following implication:

$$\|u^* - u_{2n}\| \leq (\mathcal{R}_{u_0} + \varrho)\, \delta_{\mathcal{W}}^{-1} \left(\frac{(I-\vartheta)^{2n}(\mathcal{R}_{u_0})}{\varrho} \right);$$

(iii) a posteriori error estimate holds in the following implication:

$$\|u_{2n} - u^*\| \leq (\mathcal{R}_{u_{2n}} + \varrho)\, \delta_{\mathcal{W}}^{-1} \left(\frac{\mathcal{R}_{u_{2n}})}{\varrho} \right);$$

where for all $n \geq 0$

$$\mathcal{R}_{u_0} := \vartheta^{-1}(\varrho(u_0, \mathcal{G}^2u_0)) \quad and \quad \mathcal{R}_{u_{2n}} = (I-\varphi)^{-1}(d(u_{2m}, \mathcal{G}^2u_{2m})).$$

Proof. This is a direct consequence of Example 2.4 and Theorem 3.2. Note that

$$\mathcal{R}_{u_0} := \mathcal{R}_{u_0,x} = \vartheta^{-1}(\varrho(u_0, \mathcal{G}^2u_0) + \varrho(x,x)) = \vartheta^{-1}(\varrho(u_0, \mathcal{G}^2u_0)).$$

∎

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References

[1] Çevik, C., Özeken, Ç. C. (2024). Coupled fixed point results for new classes of functions on ordered vector metric space. *Acta Math. Hungar.*, 172(1), 1–18. doi:10.1007/s10474-024-01418-x

[2] Eldred, A., Veeramani, P. (2006). Existence and convergence of best proximity points. *J. Math. Anal. Appl.*, 323(2), 1001–1006. doi:10.1016/j.jmaa.2005.10.064

[3] Espínola, R., Fernández-León, A. (2011). On best proximity points in metric and Banach space. *Canad. J. math.*, 63(3), 533–550. doi:10.4153/CJM-2010-073-5

[4] Goebel, K., Kirk, W. A. (1990). *Topics in Metric Fixed Point Theory.* Cambridge University Press. doi:10.1017/CBO9780511526152

[5] Guo, D., Lakshmikantham, V. (1987). Coupled fixed points of nonlinear operators with applications. *Nonlinear Anal.*, 11(5), 623–632. doi:10.1016/0362-546X(87)90077-0

[6] Gupta, A., Rajput, S. S., Kaurav, P. S. (2014). Coupled best proximity point theorem in metric spaces. *Inter. J. Anal. Appl.*, 4(2), 201–215.

[7] Ilchev, A. (2018). *On an application of coupled best proximity points theorems for solving systems of linear equations.* AIP Conf. Proc. 2048, 050003. doi:10.1063/1.5082089

[8] Ilchev, A., Zlatanov, B. (2016). Error estimates for approximation of coupled best proximity points for cyclic contractive maps. *Appl. Math. Comput.*, 290, 412–425. doi:10.1016/j.amc.2016.06.025

[9] Mongkolkeha, C., Dechboon, P., Salisu, S., Khammahawong, K. (2024). *Coupled best proximity points for cyclic Kannan and Chatterjea contractions in $CAT_p(0)$ spaces*, Carpathian J. Math., 40(2), 323–341. doi:10.37193/CJM.2024.02.06

[10] Puthusseria, A., Kumar, D. R. (2024). Some coupled fixed point theorems for (ψ, ϕ)-contraction with applications to fractals. *Filomat*, 38(26), 9305–9320. doi:10.2298/FIL2426305P

[11] Safari-Hafshejani, A. (2025). On the error estimates for the sequence of successive approximations for cyclic φ-contractions in metric spaces. *Carpathian J. Math.*, 41(1), 193–204. doi:10.37193/CJM.2025.01.14

[12] Safari-Hafshejani, A., Amiri. KH. (2025). Existence, convergence and approximation of optimal pair of coupled fixed points for non-cyclic contractive mappings. *Fixed point Theory*, accepted.

[13] Suzuki, T., Kikawa, M., Vetro, C. (2009). The existence of best proximity points in metric spaces with the *UC* property. *Nonlinear Anal.*, 71, 2918–2926. doi:10.1016/j.na.2009.01.180

[14] Zlatanov, B. (2016). Error estimates for approximating best proximity points for cyclic contractive maps. *Carpathian J. Math.*, 32(2), 265–270. doi:10.37193/CJM.2016.02.11

[15] Zlatanov, B. (2021). Coupled best proximity points for cyclic contractive maps and their applications. *Fixed Point Theory*, 22(1), 438–452. doi:10.24193/fpt-ro.2021.1.29