

# On the Solution of a High-Order Caputo Fractional Multi-Point Boundary Value Problem

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## Abstract

**Abstract:** This work establishes the existence and uniqueness of solutions for a nonlinear, high-order fractional differential equation with multi-point, non-local boundary conditions. The problem involves the Caputo fractional derivative. By deriving the associated Green's function, the boundary value problem is transformed into an equivalent integral equation. We then employ the Banach contraction mapping principle as the primary tool to prove our main existence and uniqueness result. The theoretical findings are supported by a numerical algorithm and a detailed illustrative example.

**Keywords:** High-order fractional differential equation; Caputo fractional derivative; Multi-point boundary value problem; Green's function; Existence and uniqueness; Fixed point theorem.

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## 1 Introduction

Fractional calculus, the study of derivatives and integrals of arbitrary real or complex order, provides powerful tools for modeling complex systems with memory and hereditary properties. Its applications are widespread, appearing in fields such as anomalous diffusion, viscoelasticity, control theory, and bioengineering [1, 2, 3, 4, 5, 6, 7, 8, 9].

Boundary value problems (BVPs) with non-local conditions have garnered significant interest because they allow for more realistic modeling of physical phenomena where measurements at the boundary depend on interior values of the domain. In particular, multi-point BVPs are essential in various engineering and physics applications [10, 11, 12]. The study of such problems in the fractional setting combines the non-local nature of the fractional operator with the non-local structure of the boundary conditions.

The primary analytical tool for proving the existence and uniqueness of solutions to nonlinear BVPs is fixed point theory. By reformulating a BVP as an integral equation via its Green's function,

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theorems such as the Banach contraction principle or Krasnoselskii's fixed point theorem can be applied to great effect [13, 14, 15, 16, 17].

This paper extends previous work on three-point fractional BVPs [18] to a more general multi-point scenario. We investigate the existence and uniqueness of a solution for the following high-order fractional BVP:

$$\begin{cases} {}_cD_t^\vartheta w(t) + g(t, w(t)) = 0, & t \in (0, 1), \\ w(0) = w'(0) = \dots = w^{(m-2)}(0) = 0, \\ w(1) = \gamma_1 w(\eta_1) + \gamma_2 w(\eta_2), \end{cases} \quad (1)$$

where  $m - 1 < \vartheta \leq m$  for an integer  $m \geq 2$ ,  $0 < \eta_1 < \eta_2 < 1$ ,  $\gamma_1, \gamma_2 \in \mathbb{R}$ ,  ${}_cD_t^\vartheta$  is the Caputo fractional derivative, and  $g \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ .

The paper is structured as follows. In Section 2, we derive the Green's function for the corresponding linear problem. Section 3 provides a key estimate for this Green's function. In Section 4, we prove our main existence and uniqueness theorem. Section 5 presents a numerical method, and Section 6 contains an illustrative example. Section 7 concludes the paper.

## 2 Formulation of the Green's function

Our first step is to analyze the linear version of the problem (1):

$$\begin{cases} {}_cD_t^\vartheta w(t) + h(t) = 0, & t \in (0, 1), \\ w(0) = w'(0) = \dots = w^{(m-2)}(0) = 0, \\ w(1) = \gamma_1 w(\eta_1) + \gamma_2 w(\eta_2), \end{cases} \quad (2)$$

where  $h \in C[0, 1]$ . We will build the solution by first considering a simpler two-point BVP.

**Lemma 2.1.** *For  $h \in C[0, 1]$ , the unique solution to the two-point BVP*

$$\begin{cases} {}_cD_t^\vartheta u(t) + h(t) = 0, & t \in (0, 1), \\ u(0) = u'(0) = \dots = u^{(m-2)}(0) = 0, & u(1) = 0, \end{cases} \quad (3)$$

is given by  $u(t) = \int_0^1 R(t, s)h(s)ds$ , where

$$R(t, s) = \frac{1}{\Gamma(\vartheta)} \begin{cases} t^{m-1}(1-s)^{\vartheta-1} - (t-s)^{\vartheta-1}, & 0 \leq s \leq t \leq 1, \\ t^{m-1}(1-s)^{\vartheta-1}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (4)$$

*Proof.* This is a standard result. The general solution of  ${}_cD_t^\vartheta u(t) = -h(t)$  is

$$u(t) = \sum_{j=0}^{m-1} c_j t^j - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s) ds.$$

The conditions  $u(0) = u'(0) = \dots = u^{(m-2)}(0) = 0$  imply  $c_0 = c_1 = \dots = c_{m-2} = 0$ . The condition  $u(1) = 0$  gives  $c_{m-1} = \frac{1}{\Gamma(\vartheta)} \int_0^1 (1-s)^{\vartheta-1} h(s) ds$ . Substituting  $c_{m-1}$  and rearranging the integrals yields the expression for  $R(t, s)$ . ■

**Theorem 2.2.** *Let  $h \in C[0, 1]$  and assume  $\Delta = 1 - \gamma_1\eta_1^{m-1} - \gamma_2\eta_2^{m-1} \neq 0$ . Then the multi-point BVP (2) has a unique solution*

$$w(t) = \int_0^1 G(t, s)h(s)ds, \tag{5}$$

where

$$G(t, s) = R(t, s) + \frac{t^{m-1}}{\Delta} [\gamma_1 R(\eta_1, s) + \gamma_2 R(\eta_2, s)]. \tag{6}$$

*Proof.* Let  $w(t)$  be the solution to (2). The general solution to  ${}_cD_t^\vartheta w(t) = -h(t)$  satisfying  $w(0) = \dots = w^{(m-2)}(0) = 0$  is

$$w(t) = c_{m-1}t^{m-1} - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s)ds.$$

Applying the multi-point condition  $w(1) = \gamma_1 w(\eta_1) + \gamma_2 w(\eta_2)$ , we get

$$\begin{aligned} c_{m-1} - I^\vartheta h(1) &= \gamma_1 \left( c_{m-1} \eta_1^{m-1} - I^\vartheta h(\eta_1) \right) \\ &\quad + \gamma_2 \left( c_{m-1} \eta_2^{m-1} - I^\vartheta h(\eta_2) \right), \end{aligned}$$

where  $I^\vartheta h(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s)ds$ . Solving for  $c_{m-1}$ :

$$c_{m-1}(1 - \gamma_1\eta_1^{m-1} - \gamma_2\eta_2^{m-1}) = I^\vartheta h(1) - \gamma_1 I^\vartheta h(\eta_1) - \gamma_2 I^\vartheta h(\eta_2).$$

Since  $\Delta \neq 0$ ,

$$c_{m-1} = \frac{1}{\Delta} \left[ I^\vartheta h(1) - \gamma_1 I^\vartheta h(\eta_1) - \gamma_2 I^\vartheta h(\eta_2) \right].$$

The solution  $w(t)$  can be written in terms of  $u(t)$  from Lemma 2.1 as  $w(t) = u(t) + \lambda t^{m-1}$  for some constant  $\lambda$ . The boundary conditions  $w(0) = \dots = w^{(m-2)}(0) = 0$  are satisfied since  $u(t)$  satisfies them. The final condition gives  $u(1) + \lambda = \gamma_1(u(\eta_1) + \lambda\eta_1^{m-1}) + \gamma_2(u(\eta_2) + \lambda\eta_2^{m-1})$ . Since  $u(1) = 0$ , we solve for  $\lambda$ :

$$\begin{aligned} \lambda(1 - \gamma_1\eta_1^{m-1} - \gamma_2\eta_2^{m-1}) &= \gamma_1 u(\eta_1) + \gamma_2 u(\eta_2), \\ \lambda &= \frac{\gamma_1 u(\eta_1) + \gamma_2 u(\eta_2)}{\Delta}. \end{aligned}$$

Thus,  $w(t) = u(t) + \frac{t^{m-1}}{\Delta}(\gamma_1 u(\eta_1) + \gamma_2 u(\eta_2))$ . Substituting the integral form of  $u(t)$  yields the desired result for  $G(t, s)$ . ■

### 3 Estimation of the Green's function

To prove our main theorem, we need an upper bound for the integral of  $|G(t, s)|$ .

**Lemma 3.1.** *Let  $R(t, s)$  be the Green's function from Lemma 2.1. Then for all  $t \in [0, 1]$ ,*

$$\int_0^1 |R(t, s)|ds \leq \frac{2}{\Gamma(\vartheta + 1)}. \tag{7}$$

*Proof.* By the triangle inequality,

$$\begin{aligned} \int_0^1 |R(t, s)| ds &= \int_0^t \left| \frac{t^{m-1}(1-s)^{\vartheta-1} - (t-s)^{\vartheta-1}}{\Gamma(\vartheta)} \right| ds + \int_t^1 \left| \frac{t^{m-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} \right| ds \\ &\leq \frac{1}{\Gamma(\vartheta)} \left( \int_0^t t^{m-1}(1-s)^{\vartheta-1} ds + \int_0^t (t-s)^{\vartheta-1} ds + \int_t^1 t^{m-1}(1-s)^{\vartheta-1} ds \right) \\ &= \frac{1}{\Gamma(\vartheta)} \left( t^{m-1} \int_0^1 (1-s)^{\vartheta-1} ds + \int_0^t (t-s)^{\vartheta-1} ds \right) \\ &= \frac{1}{\Gamma(\vartheta)} \left( t^{m-1} \frac{1}{\vartheta} + \frac{t^\vartheta}{\vartheta} \right) = \frac{t^{m-1} + t^\vartheta}{\Gamma(\vartheta + 1)}. \end{aligned}$$

Since  $t \in [0, 1]$  and  $m - 1 < \vartheta$ , we have  $t^{m-1} \leq 1$  and  $t^\vartheta \leq 1$ . Therefore,  $\int_0^1 |R(t, s)| ds \leq \frac{1+1}{\Gamma(\vartheta+1)} = \frac{2}{\Gamma(\vartheta+1)}$ . ■

**Theorem 3.2.** *Let  $G(t, s)$  be the Green's function from Theorem 2.2. Then for all  $t \in [0, 1]$ ,*

$$\int_0^1 |G(t, s)| ds \leq \frac{2}{\Gamma(\vartheta + 1)} \left( 1 + \frac{|\gamma_1| + |\gamma_2|}{|\Delta|} \right). \quad (8)$$

*Proof.* Using the triangle inequality on the definition of  $G(t, s)$ :

$$\begin{aligned} \int_0^1 |G(t, s)| ds &\leq \int_0^1 |R(t, s)| ds + \int_0^1 \left| \frac{t^{m-1}}{\Delta} [\gamma_1 R(\eta_1, s) + \gamma_2 R(\eta_2, s)] \right| ds \\ &\leq \int_0^1 |R(t, s)| ds + \frac{|t^{m-1}|}{|\Delta|} \left( |\gamma_1| \int_0^1 |R(\eta_1, s)| ds + |\gamma_2| \int_0^1 |R(\eta_2, s)| ds \right). \end{aligned}$$

Since  $t \in [0, 1]$ , we have  $|t^{m-1}| \leq 1$ . Applying Lemma 3.1 to each integral gives

$$\begin{aligned} \int_0^1 |G(t, s)| ds &\leq \frac{2}{\Gamma(\vartheta + 1)} + \frac{1}{|\Delta|} \left( |\gamma_1| \frac{2}{\Gamma(\vartheta + 1)} + |\gamma_2| \frac{2}{\Gamma(\vartheta + 1)} \right) \\ &= \frac{2}{\Gamma(\vartheta + 1)} \left( 1 + \frac{|\gamma_1| + |\gamma_2|}{|\Delta|} \right). \end{aligned}$$

This completes the proof. ■

## 4 Existence and uniqueness of the solution

We are now ready to state and prove our main result.

**Theorem 4.1.** *Let  $g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function that satisfies a uniform Lipschitz condition with respect to its second argument. That is, there exists a constant  $L > 0$  such that for all  $t \in [0, 1]$  and  $w, z \in \mathbb{R}$ ,*

$$|g(t, w) - g(t, z)| \leq L|w - z|. \quad (9)$$

*If  $\Delta = 1 - \gamma_1 \eta_1^{m-1} - \gamma_2 \eta_2^{m-1} \neq 0$  and the Lipschitz constant  $L$  satisfies*

$$\frac{2L}{\Gamma(\vartheta + 1)} \left( 1 + \frac{|\gamma_1| + |\gamma_2|}{|\Delta|} \right) < 1, \quad (10)$$

*then the fractional multi-point BVP (1) has a unique solution in  $C[0, 1]$ .*

*Proof.* Let  $X = C[0, 1]$  be the Banach space of continuous functions on  $[0, 1]$  equipped with the supremum norm  $\|w\| = \max_{t \in [0, 1]} |w(t)|$ . We define an operator  $T : X \rightarrow X$  based on the integral equation equivalent to (1):

$$(Tw)(t) = \int_0^1 G(t, s)g(s, w(s))ds, \quad (11)$$

where  $G(t, s)$  is given by (6). A fixed point of  $T$  is a solution to our BVP. We will show that  $T$  is a contraction mapping.

Let  $w, z \in X$ . Consider

$$\begin{aligned} |(Tw)(t) - (Tz)(t)| &= \left| \int_0^1 G(t, s)[g(s, w(s)) - g(s, z(s))]ds \right| \\ &\leq \int_0^1 |G(t, s)||g(s, w(s)) - g(s, z(s))|ds \\ &\leq \int_0^1 |G(t, s)|L|w(s) - z(s)|ds \\ &\leq L\|w - z\| \int_0^1 |G(t, s)|ds. \end{aligned}$$

Taking the maximum over  $t \in [0, 1]$  and applying Theorem 3.2, we get

$$\begin{aligned} \|Tw - Tz\| &\leq L\|w - z\| \max_{t \in [0, 1]} \int_0^1 |G(t, s)|ds \\ &\leq \|w - z\| \left[ \frac{2L}{\Gamma(\vartheta + 1)} \left( 1 + \frac{|\gamma_1| + |\gamma_2|}{|\Delta|} \right) \right]. \end{aligned}$$

Let  $\psi = \frac{2L}{\Gamma(\vartheta + 1)} \left( 1 + \frac{|\gamma_1| + |\gamma_2|}{|\Delta|} \right)$ . The condition (10) means that  $\psi < 1$ . Therefore, we have  $\|Tw - Tz\| \leq \psi\|w - z\|$  with  $\psi < 1$ , which shows that  $T$  is a contraction mapping on the complete metric space  $X$ . By the Banach contraction mapping principle,  $T$  has a unique fixed point, which is the unique solution to the BVP (1). ■

## 5 Numerical algorithm

The proof of Theorem 4.1 is constructive and naturally suggests an iterative method for approximating the unique solution. The iteration scheme, known as Picard's iteration, is given by

$$w_{k+1}(t) = \int_0^1 G(t, s)g(s, w_k(s))ds, \quad k = 0, 1, 2, \dots \quad (12)$$

Starting with an initial guess  $w_0(t)$  (for example,  $w_0(t) \equiv 0$ ), the sequence  $\{w_k(t)\}$  converges uniformly to the unique solution  $w(t)$  of the BVP (1). The iteration can be terminated when  $\|w_{k+1} - w_k\| < \varepsilon$  for a prescribed tolerance  $\varepsilon > 0$ .

While direct symbolic computation of the integrals in (12) can be slow, this scheme forms the basis for more advanced numerical methods. For instance, the integral can be approximated using a numerical quadrature rule, and the function  $w_k(s)$  can be represented using interpolation on a discrete grid, leading to a fully numerical solver.

## 6 Illustrative Example

Consider the following high-order fractional multi-point BVP:

$$\begin{cases} {}_c D_t^{2.5} w(t) + \beta \arctan(w(t)) + e^{-t^2} = 0, & t \in (0, 1), \beta > 0 \\ w(0) = w'(0) = 0, \\ w(1) = \frac{1}{2}w(\frac{1}{3}) + \frac{1}{4}w(\frac{2}{3}). \end{cases} \quad (13)$$

Here,  $\vartheta = 2.5$ , so  $m = 3$ . The parameters are  $\gamma_1 = 1/2$ ,  $\eta_1 = 1/3$ ,  $\gamma_2 = 1/4$ , and  $\eta_2 = 2/3$ . The nonlinear function is  $g(t, w) = \beta \arctan(w) + e^{-t^2}$ .

First, we check the condition on  $\Delta$ :

$$\Delta = 1 - \gamma_1 \eta_1^{m-1} - \gamma_2 \eta_2^{m-1} = 1 - \frac{1}{2} \left(\frac{1}{3}\right)^2 - \frac{1}{4} \left(\frac{2}{3}\right)^2 = 1 - \frac{1}{18} - \frac{1}{9} = \frac{15}{18} = \frac{5}{6} \neq 0.$$

The function  $g(t, w)$  is Lipschitz with respect to  $w$ . Using the mean value theorem, for any  $w, z \in \mathbb{R}$ :

$$|g(t, w) - g(t, z)| = |\beta \arctan(w) - \beta \arctan(z)| = \beta \left| \frac{1}{1+c^2}(w-z) \right| \leq \beta |w-z|,$$

where  $c$  is between  $w$  and  $z$ . The Lipschitz constant is  $L = \beta$ .

Now, we apply the main condition from Theorem 4.1:

$$\frac{2\beta}{\Gamma(2.5+1)} \left( 1 + \frac{|1/2| + |1/4|}{|5/6|} \right) < 1.$$

We calculate the constants:

- $\Gamma(3.5) = \Gamma(2.5+1) = 2.5 \cdot 1.5 \cdot 0.5 \cdot \sqrt{\pi} = \frac{15}{8} \sqrt{\pi} \approx 3.32335$ .
- $1 + \frac{|1/2|+|1/4|}{|5/6|} = 1 + \frac{3/4}{5/6} = 1 + \frac{18}{20} = 1 + \frac{9}{10} = \frac{19}{10} = 1.9$ .

The inequality becomes:

$$\frac{2\beta}{3.32335} \times 1.9 < 1$$

$$\beta \left( \frac{3.8}{3.32335} \right) < 1 \implies \beta(1.14342) < 1 \implies \beta < \frac{1}{1.14342} \approx 0.87457.$$

Thus, according to Theorem 4.1, if  $0 < \beta < 0.87457$ , the BVP (13) is guaranteed to have a unique solution in  $C[0, 1]$ .

## 7 Conclusion

In this paper, we have successfully established sufficient conditions for the existence and uniqueness of solutions to a class of high-order nonlinear fractional differential equations subject to multi-point, non-local boundary conditions. The methodology relied on constructing an appropriate Green's function to reformulate the problem as an integral equation, followed by an application of the Banach contraction mapping principle. The derived conditions provide a clear and practical criterion for ensuring a unique solution, as demonstrated by our illustrative example. This work contributes to the growing body of literature on non-local fractional boundary value problems and provides a solid foundation for further numerical and analytical investigations.

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