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Research Article

Some fixed point theorems for $(\alpha, \psi, \mathcal{F}, h)$-rational-1-1-upclass type contractive mappings

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Abstract

Abstract: In this paper, we via pair 1-1-upclass function introduce the new concept of $(\alpha, \psi, \mathcal{F}, h)$-rational 1-1-upclass contractive mappings and obtain some new results with several interesting corollaries. Also, we state an example for support main result.

Keywords: Fixed point, Generalized metric space, α-Admissible mapping, α-Regular, μ-Subadmissible mapping, μ-Subregular, 1-1-Upclass function.
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1 Introduction and Preliminaries

Fixed point theory has gained very large impetus due to its wide range of applications in various fields such as engineering, economics, computer science, and many others. It is well known that the contractive conditions are very indispensable in the study of fixed point theory and Banach's fixed point theorem [1] for contraction mappings is one of the pivotal result in analysis. This theorem has been extended and generalized by various authors (see [2]) in various abstract spaces one of which

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is generalized metric space. A short paper for history of survey of the development of fixed point theory can see in [7].

In this paper, we via pair 1-1-upclass function introduce the new concept of $(\alpha, \psi, \mathcal{F}, h)$-rational-1-1-upclass contractive mappings and obtain some new results in [29, 34] with several interesting corollaries. We also state one example for support main result.

Now, we give some notations and introduce some definitions which will be used in the sequel.

Definition 1.1. [2] Let X be a nonempty set and a mapping $d : X \times X \to [0, +\infty)$ for all $x, y \in X$ and all distinct $u, v \in X \backslash \{x, y\}$ satisfy the following conditions:

(GMS1) $d(x, y) = 0$ if and only if $x = y$,

(GMS2) $d(x, y) = d(y, x)$,

(GMS3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then the mapping d is called a generalized metric and abbreviated as GM. Here, the pair (X, d) is called a generalized metric space and abbreviated as GMS.

In the above definition, if d satisfies only (GMS1) and (GMS2), then it is called a semimetric (see [19]).

A sequence $\{x_n\}$ in a GMS (X, d) is GMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to +\infty$.

A sequence $\{x_n\}$ in a GMS (X, d) is GMS Cauchy if and only if for every $\epsilon > 0$ there exists positive integer $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$, for all $n > m > N(\epsilon)$.

A GMS (X, d) is called complete if every GMS Cauchy sequence in X is GMS convergent.

A mapping $T : X \to X$ is a continuous if for any sequence $\{x_n\}$ in X such that $d(x_n, x) \to 0$ as $n \to +\infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to +\infty$.

The following assumption was suggested by Wilson [19] to replace the triangle inequality with the weakened condition.

(W) For each pair of (distinct) points u, v, there is a number $r_{u,v} > 0$ such that for every $z \in X$, $r_{u,v} < d(u, z) + d(z, v)$.

Definition 1.2. Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ be two mappings. We say that T is an α-admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(Tx,\ Ty) \geq 1$, for all $x, y \in X$.

Definition 1.3. [29] Let (X, d) be a GMS and $\alpha : X \times X \to [0, +\infty)$. The set X is called α-regular GMS if, for a sequence $\{x_n\}$ in X such that $x_n \to x$ and $\alpha(x_n,\ x_{n+1}) \geq\ 1$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$, for all $k \in \mathbb{N}$.

Definition 1.4. Let X be a nonempty set, $T : X \to X$ and $\mu : X \times X \to [0, +\infty)$ be two mappings. We say that T is an μ-subadmissible mapping if $\mu(x, y) \leq 1$ implies $\mu(Tx,\ Ty) \leq 1$, for all $x, y \in X$.

Definition 1.5. Let (X, d) be a GMS and $\mu : X \times X \to [0, +\infty)$. The set X is called μ-subregular GMS if, for a sequence $\{x_n\}$ in X such that $x_n \to x$ and $\mu(x_n,\ x_{n+1}) \leq\ 1$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\mu(x_{n_k}, x) \leq\ 1$ for all $k \in \mathbb{N}$.

Definition 1.6. [31] A mapping $h : [0, +\infty) \to [0, +\infty)$ is an A-class function if $h(t) \geq t$, for all $t \geq 0$. With $\mathcal{A}$ we denote the set of all $\mathcal{A}$-class functions.

Example 1.7. [31] The following functions $h : [0, +\infty) \to [0, +\infty)$ are elements of $\mathcal{A}$

(1) $h(t) = a^t - 1$, $a > 1$, $t \in [0, +\infty)$,

(2) $h(t) = mt$, $m \geq 1$, $t \in [0, +\infty)$.

Definition 1.8. [31] A mapping $\mathcal{F} : [0, +\infty)^4 \to \mathbb{R}$ is a 1-1-upclass function if for all $u, v, s, t \in [0, +\infty)$ the following conditions hold:

1. $\mathcal{F}(1, 1, s, t)$ is continuous;
2. $0 \leq u \leq 1, v \geq 1 \Rightarrow \mathcal{F}(u, v, s, t) \leq \mathcal{F}(1, 1, s, t) \leq s$;
3. $\mathcal{F}(1, 1, s, t) = s \Rightarrow s = 0$ or $t = 0$.

We denote $\mathcal{C}$ the set of all 1-1-upclass functions.

Remark 1.9. [31] Note that $\mathcal{F}(1, 1, 0, 0) = 0$.

Example 1.10. [31] The following functions $\mathcal{F} : [0, +\infty)^4 \to \mathbb{R}$ are elements of $\mathcal{C}$ for all $u, v, s, t \in [0, +\infty)$:

1. $\mathcal{F}(u, v, s, t) = us - vt$, $\mathcal{F}(1, 1, s, t) = s \Rightarrow t = 0$;
2. $\mathcal{F}(u, v, s, t) = us - vt$, $\mathcal{F}(1, 1, s, t) = s \Rightarrow t = 0$;
3. $\mathcal{F}(u, v, s, t) = \dfrac{us - vt}{1 + vt}$, $\mathcal{F}(1, 1, s, t) = s \Rightarrow t = 0$;
4. $f(u, v, s, t) = \dfrac{us}{1 + vt}$, $f(1, 1, s, t) = s \Rightarrow s = 0$ or $t = 0$;
5. $\mathcal{F}(u, v, s, t) = \log_a \dfrac{ut + a^{us}}{1 + vt}$, $a > 1$, $\mathcal{F}(1, 1, s, t) = s \Rightarrow s = 0$ or $t = 0$;
6. $\mathcal{F}(u, v, s, t) = \ln \dfrac{u + e^{us}}{1 + v}$,$\mathcal{F}(1, 1, s, 1) = s \Rightarrow s = 0$;
7. $\mathcal{F}(u, v, s, t) = (us + a)^{\frac{1}{1+vt}} - a$, $a > 1$, $\mathcal{F}(1, 1, s, t) = s \Rightarrow t = 0$;
8. $\mathcal{F}(u, v, s, t) = us \log_{a+vt} a$, $a > 1$, $\mathcal{F}(1, 1, s, t) = s \Rightarrow s = 0$ or $t = 0$

Definition 1.11. [32] A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote the set of altering distance functions by Φ.

Definition 1.12. [30] Let Φ_u denote the class of the functions $\varphi : [0, +\infty) \to [0, +\infty)$ which satisfy the following conditions:

(i) φ is continuous;

(ii) $\varphi(t) > 0$, for all $t > 0$ and $\varphi(0) \geq 0$.

Throughout the paper, $F(T)$ denotes the set of fixed points of the mapping T.

Proposition 1.13. *[21] In a semimetric space, the assumption (W) is equivalent to the assertion that the limits are unique.*

Proposition 1.14. *[21]Suppose that $\{x_n\}$ is a Cauchy sequence in a $GMS(X,d)$ with $lim_{n\to+\infty}d(x_n,u) = 0$, where $u \in X$. Then $lim_{n\to+\infty}d(x_n,z) = d(u,z)$, for all $z \in X$. In particular,the sequence $\{x_n\}$ does not converge to z if $z \neq u$.*

Lemma 1.15. *[33, Lemma 1] Let (X,d) be a generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ can converge to at most one point.*

Lemma 1.16. *[33] Let (X,d) be a generalized metric space and let $\{x_n\}$ be a sequence in X with distinct elements ($x_n \neq x_m$ for $n \neq m$). Suppose that $d(x_n,x_{n+1})$ and $d(x_n,x_{n+2})$ tends to 0 as $n \to +\infty$ and that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences*

$$d(x_{m_k},x_{n_k}), \quad d(x_{m_k},x_{n_{k+1}}), \quad d(x_{m_{k-1}},x_{n_k}), \quad d(x_{m_{k-1}},x_{n_{k+1}}) \tag{1}$$

tends to ϵ as $k \to +\infty$.

2 Main Results

The contraction mappings considered in this paper are constructed via auxiliary functions defined below. Let Ψ be a family of functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfying the following properties

(i) ψ is upper semi-continuous, strictly increasing;

(ii) $\{\psi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to +\infty$, for all $t > 0$;

(iii) $\psi(t) < t$, for every $t > 0$.

Definition 2.1. Let (X,d) be a GMS and $\alpha,\mu : X \times X \to [0,+\infty)$. A self mapping $T : X \to X$ is said to be $(\alpha,\psi,\mathcal{F},h)$-rational-1-1-upclass contractive mapping if for all $x,y \in X$ the following condition holds

$$h(\psi(d(Tx,Ty))) \leq \mathcal{F}(\mu(x,y),\alpha(x,y),\psi((M(x,y)),\varphi(M(x,y)))), \tag{2}$$

where $\psi \in \Phi, \varphi \in \Phi_u$, $\mathcal{F}$ is a 1-1-upclass and h is a A-class function and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\}.$$

Next, we prove existence and uniqueness for fixed point of $(\alpha,\psi,\mathcal{F},h)$-rational-1-1-upclass contractive mappings.

Theorem 2.2. *Let (X,d) be a complete GMS, $T : X \to X$ be a self mapping and $\alpha, \mu : X \times X \to [0,+\infty)$ a given functions. Suppose that the following conditions are satisfied:*

(i) T is an α-admissible mapping and μ-subadmissible mapping;

(ii) T is an $(\alpha, \psi, \mathcal{F}, h)$-rational-1-1-upclass contractive mapping;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, T\,x_0) \geq 1$, $\alpha(x_0, T^2\,x_0) \geq 1$ and $\mu(x_0, T\,x_0) \leq 1$, $\mu(x_0, T^2\,x_0) \leq 1$;

(iv) either T is continuous, or X is α-regular and μ-subregular.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*. Further, if for all $x, y \in F(T)$, we have $\alpha(x,y) \geq 1, \mu(x,y) \leq 1$ then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ satisfies $\alpha(x_0, T\,x_0) \geq 1$ and $\alpha(x_0, T^2\,x_0) \geq 1$. We construct the sequence $\{x_n\}$ in X as $x_n = T^n\,x_0 = T\,x_{n-1}$, for $n \in \mathbb{N}$. It is obvious that if $x_{n_0} = x_{n_0+1}$, for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T. Consequently, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since T is α-admissible we have $\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1$, this implies $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$, thus, $\alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1$. Hence by induction, we get $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$.

By similar arguments, since $\alpha(x_0, T^2\,x_0) \geq 1$, we have $\alpha(x_0, x_2) = \alpha(x_0, T^2 x_0) \geq 1$, $\alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1$. By induction, we get $\alpha(x_n, x_{n+2}) \geq 1$ for all $n \geq 0$. Also for μ, since T is μ-subadmissible then $\mu(x_0, Tx_0) = \mu(x_0, x_1) \leq 1$. This implies $\mu(Tx_0, Tx_1) = \mu(x_1, x_2) \leq 1$, thus, $\mu(Tx_1, Tx_2) = \mu(x_2, x_3) \leq 1$, hence by induction, we get $\mu(x_n, x_{n+1}) \leq 1$ for all $n \geq 0$. By similar arguments, since $\mu(x_0, T^2\,x_0) \leq 1$, we have $\mu(x_0, x_2) = \mu(x_0, T^2 x_0) \leq 1$, $\mu(Tx_0, Tx_2) = \mu(x_1, x_3) \leq 1$. By induction, we get $\mu(x_n, x_{n+2}) \leq 1$ for all $n \geq 0$.

Consider (2) with $x = x_n$ and $y = x_{n+1}$. Clearly, we have

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &\leq h(\psi(d(x_{n+1}, x_{n+2}))) = h(\psi(d(Tx_n, Tx_{n+1}))) \\ &\leq \mathcal{F}(\mu(x_n, x_{n+1}), \alpha(x_n, x_{n+1}), \psi(M(x_n, x_{n+1})), \varphi(M(x_n, x_{n+1}))) \\ &\leq \mathcal{F}(1, 1, \psi(M(x_n, x_{n+1})), \varphi(M(x_n, x_{n+1}))) \leq \psi(M(x_n, x_{n+1})).\end{aligned}$$

So, we have

$$\begin{aligned} 0 &< \psi(d(x_{n+1}, x_{n+2})) \\ &\leq \mathcal{F}(1, 1, \psi(M(x_n, x_{n+1})), \varphi(M(x_n, x_{n+1}))) \leq \psi(M(x_n, x_{n+1})), \end{aligned} \tag{3}$$

for all $n \geq 1$, where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ &\quad \frac{d(x_n, Tx_n) d(x_{n+1}, Tx_{n+1})}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, Tx_n) d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_n, Tx_{n+1})}\} \\ &= \max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ &\quad \frac{d(x_n, x_{n+1}) d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+1})}, \frac{d(x_n, x_{n+1}) d(x_{n+1}, x_{n+2})}{1 + d(x_{n+1}, x_{n+2})}\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}, \end{aligned}$$

since

$$\frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+1})} \leq d(x_{n+1}, x_{n+2})$$

and

$$\frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_{n+1}, x_{n+2})} \leq d(x_n, x_{n+1}).$$

If for some n, $M(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$, then we have

$$\begin{aligned} 0 &< \psi(d(x_{n+1}, x_{n+2})) \\ &\leq \mathcal{F}(1, 1, \psi(d(x_{n+1}, x_{n+2})), \varphi(d(x_{n+1}, x_{n+2}))) \leq \psi(d(x_{n+1}, x_{n+2})), \end{aligned}$$

We deduce that

$$\psi(d(x_{n+1}, x_{n+2})) = 0 \text{ or } \varphi(d(x_{n+1}, x_{n+2})) = 0,$$

that is $d(x_{n+1}, x_{n+2}) = 0$, which is impossible. Hence,

$$M(x_n, x_{n+1}) = d(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}$. So,

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(M(x_n, x_{n+1})) = \psi(d\,(x_n,x_{n+1})). \tag{4}$$

From the property of ψ, we conclude that

$$d\,(x_{n+1},x_{n+2}) < d\,(x_n,x_{n+1})\,, \tag{5}$$

Thus, we conclude that the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. So, there exists $r \geq 0$ such that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = r$. Now, on account of (3), we get that

$$\psi(r) \leq F(1, 1, \psi(r), \varphi(r)),$$

which yields that $\psi(r) = 0$, or $, \varphi(r) = 0$. We derive

$$r = \lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{6}$$

Analogously, we shall prove that $\lim_{n\to+\infty} d(x_n, x_{n+2}) = 0$. By substituting $x = x_{n-1}$ and $y = x_{n+1}$ in (2),

$$\begin{aligned} &\psi(d(x_n, x_{n+2})) \leq h(\psi(d(x_n, x_{n+2}))) = h(\psi(d(Tx_{n-1},Tx_{n+1}))) \\ &\leq \mathcal{F}(\mu(x_{n-1}, x_{n+1}), \alpha(x_{n-1}, x_{n+1}), \psi(M(x_{n-1}, x_{n+1})), \varphi(M(x_{n-1}, x_{n+1}))) \\ &\leq \mathcal{F}(1, 1, \psi(M(x_{n-1}, x_{n+1})), \varphi(M(x_{n-1}, x_{n+1}))) \leq \psi(M(x_{n-1}, x_{n+1})). \end{aligned}$$

This implies

$$\begin{aligned} 0 &< \psi(d(x_n, x_{n+2})) \\ &\leq \mathcal{F}(1, 1, \psi(M(x_{n-1}, x_{n+1})), \varphi(M(x_{n-1}, x_{n+1}))) \leq \psi(M(x_{n-1}, x_{n+1})), \end{aligned} \tag{7}$$

for all $n \geq 1$, where

$$\begin{aligned} & M(x_{n-1}, x_{n+1}) = \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1}), \\ & \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(x_{n-1}, x_{n+1})}, \frac{d(x_{n-1}, Tx_{n-1})d(x_{n+1}, Tx_{n+1})}{1 + d(Tx_{n-1}, Tx_{n+1})}\Bigg\} \\ = \; & \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), \\ & \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_{n-1}, x_{n+1})}, \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+2})}\} \\ \leq \; & \max\{d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), \\ & \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_{n-1}, x_{n+1})}, \frac{d(x_{n-1}, x_n)d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+2})}\} \\ & \text{by } (6) \\ \to \; & d(x_n, x_{n+2}) \end{aligned}$$

Now, from (3) we get that

$$\begin{aligned} \psi(\lim_{n\to+\infty} d(x_n, x_{n+2})) & \leq F(\psi(\lim_{n\to+\infty} d(x_n, x_{n+2})), \varphi(\lim_{n\to+\infty} d(x_n, x_{n+2}))) \\ & \leq \psi(\lim_{n\to+\infty} d(x_n, x_{n+2})), \end{aligned}$$

which yields that $\psi(\lim_{n\to+\infty} d(x_n, x_{n+2})) = 0$ or $\varphi(\lim_{n\to+\infty} d(x_n, x_{n+2})) = 0$. We derive

$$\lim_{n\to+\infty} d(x_n, x_{n+2}) = 0. \tag{8}$$

We are ready to prove that $\{x_n\}$ is a Cauchy sequence in (X, d). Otherwise, by Lemma 1.16 there exists an $\varepsilon > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)}, x_{2n(k)-2}) < \varepsilon$ and

$$\begin{aligned} \lim_{k\to+\infty} d\left(x_{n(k)}, x_{m(k)}\right) & = \lim_{k\to+\infty} d\left(x_{n(k)+1}, x_{m(k)}\right) \\ & = \lim_{k\to+\infty} d\left(x_{n(k)}, x_{m(k)-1}\right) \\ & = \lim_{k\to+\infty} d\left(x_{n(k)+1}, x_{m(k)+1}\right) = \varepsilon. \end{aligned} \tag{9}$$

By substituting $x = x_{n(k)}$ and $y = x_{m(k)}$ in (2)

$$\begin{aligned} & \psi(d(x_{n(k)+1}, x_{m(k)+1})) \leq h(\psi(d(x_{n(k)+1}, x_{m(k)+1}))) = h(\psi(d(Tx_{n(k)}, Tx_{m(k)}))) \\ & \leq \mathcal{F}(\mu(x_{n(k)}, x_{m(k)}), \alpha(x_{n(k)}, x_{m(k)}), \psi(M(x_{n(k)}, x_{m(k)})), \varphi(M(x_{n(k)}, x_{m(k)}))) \\ & \leq \mathcal{F}(1, 1, \psi(M(x_{n(k)}, x_{m(k)})), \varphi(M(x_{n(k)}, x_{m(k)}))) \leq \psi(M(x_{n(k)}, x_{m(k)})). \end{aligned}$$

This implies

$$\begin{aligned} 0 & < \psi(d(x_{n(k)+1}, x_{m(k)+1})) \\ & \leq \mathcal{F}(1, 1, \psi(M(x_{n(k)}, x_{m(k)})), \varphi(M(x_{n(k)}, x_{m(k)}))) \leq \psi(M(x_{n(k)}, x_{m(k)})), \end{aligned} \tag{10}$$

for all $n \geq 1$, where

$$\begin{aligned}
&M(x_{n(k)}, x_{m(k)}) = \max\{d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, Tx_{n(k)}), d(x_{m(k)}, Tx_{m(k)}),\\
&\quad \frac{d(x_{n(k)}, Tx_{n(k)})d(x_{m(k)}, Tx_{m(k)})}{1+d(x_{n(k)}, x_{m(k)})}, \frac{d(x_{n(k)}, Tx_{n(k)})d(x_{m(k)}, Tx_{m(k)})}{1+d(Tx_{n(k)}, Tx_{m(k)})}\}\\
&= \max\{d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1}),\\
&\quad \frac{d(x_{n(k)}, x_{n(k)+1})d(x_{m(k)}, x_{m(k)+1})}{1+d(x_{n(k)}, x_{m(k)})}, \frac{d(x_{n(k)}, x_{n(k)+1})d(x_{m(k)}, x_{m(k)+1})}{1+d(x_{n(k)+1}, x_{m(k)+1})}\}\\
&\to \varepsilon,
\end{aligned}$$

Letting $k \to +\infty$, now, use (6) we deduce that

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)),$$

which implies $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$. Consequently, we get $\varepsilon = 0$, which is a contradiction. Hence we conclude that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to+\infty} d(x_n, x^*) = 0. \tag{11}$$

We will show next that the limit x^* of the sequence $\{x_n\}$ is a fixed point of T. First, we suppose that T is continuous. Then from (11) we have

$$\lim_{n\to+\infty} d(Tx_n, Tx^*) = \lim_{n\to+\infty} d(x_{n+1}, Tx^*) = 0. \tag{12}$$

Due to Proposition 1.14, we conclude that $x^* = Tx^*$, that is, x^* is a fixed point of T.

Now, we suppose that X is α-regular and μ-subregular. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k-1}, x^*) \geq 1, \mu(x_{n_k-1}, x^*) \leq 1$ for all $k \in \mathbb{N}$. Now, from inequality (2) with $x = x_{n_k}$ and $y = x^*$, we obtain

$$\begin{aligned}
\psi(d(x_{n_k+1}, Tx^*)) &\leq h(\psi(d(x_{n_k+1}, Tx^*))) = h(\psi(d(T\ x_{n_k}, Tx^*)))\\
&\leq \mathcal{F}(\mu(x_{n_k}, x^*), \alpha(x_{n_k}, x^*), \psi(M(x_{n_k}, x^*)), \varphi(M(x_{n_k}, x^*)))\\
&\leq \mathcal{F}(1, 1, \psi(M(x_{n_k}, x^*)), \varphi(M(x_{n_k}, x^*))) \leq \psi(M(x_{n_k}, x^*))\\
&\Longrightarrow
\end{aligned}$$

$$\begin{aligned}
0 &< \psi(d(x_{n_k+1}, Tx^*))\\
&\leq \mathcal{F}(1, 1, \psi(M(x_{n_k}, x^*)), \varphi(M(x_{n_k}, x^*))) \leq \psi(M(x_{n_k}, x^*)),
\end{aligned} \tag{13}$$

for all $n \geq 1$, where

$$\begin{aligned}
M(x_{n_k}, x^*) &= \max\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*),\\
&\quad \frac{d(x_{n_k}, Tx_{n_k})d(x^*, Tx^*)}{1+d(x_{n_k}, x^*)}, \frac{d(x_{n_k}, Tx_{n_k})d(x^*, Tx^*)}{1+d(Tx_{n_k}, Tx^*)}\}\\
&= \max\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*),\\
&\quad \frac{d(x_{n_k}, x_{n_k+1})d(x^*, Tx^*)}{1+d(x_{n_k}, x^*)}, \frac{d(x_{n_k}, x_{n_k+1})d(x^*, Tx^*)}{1+d(x_{n_k+1}, Tx^*)}\}.
\end{aligned} \tag{14}$$

Letting $k \to +\infty$ in (12) , we obtain $M(x_{n_k}, x^*) = d(x^*, Tx^*)$. Therefore, upon taking limit as $k \to +\infty$, in inequality (13), we have

$$\psi(d(x^*, Tx^*)) \leq F(\psi(d(x^*, Tx^*)), \varphi(d(x^*, Tx^*))),$$

which implies $\psi(d(x^*, Tx^*)) = 0$ or $\varphi\left(d(x^*, Tx^*)\right) = 0$,so, $x^* = Tx^*$, that is, x^* is a fixed point of T.

Finally, suppose that x^* and y^* are two fixed points of T such that $x^* \neq y^*$. Then by the hypothesis, $\alpha(x^*, y^*) \geq 1, \mu(x^*, y^*) \leq 1$. Hence, from (2) with $x = x^*$ and $y = y^*$ we have,

$$\begin{aligned}\psi(d(\ x^*, y^*)) &\leq h(\psi(d(x^*, y^*))) = h(\psi(d(Tx^*\ , Ty^*)))\\ &\leq \mathcal{F}(\mu(x^*, y^*), \alpha(x^*, y^*), \psi(M(x^*, y^*)), \varphi(M(x^*, y^*)))\\ &\leq \mathcal{F}(1, 1, \psi(M(x^*, y^*)), \varphi(M(x^*, y^*))) \leq \psi(M(x^*, y^*))\\ &\Longrightarrow\end{aligned}$$

$$\begin{aligned}0 &< \psi(d(\ \ x^*, y^*))\\ &\leq \mathcal{F}(1, 1, \psi(M(x^*, y^*)), \varphi(M(x^*, y^*))) \leq \psi(M(x^*, y^*)).\end{aligned} \tag{15}$$

where

$$\begin{aligned}M(x^*, y^*) &= \max\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*),\\ &\quad \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{1 + d(x^*, y^*)}, \frac{d(x^*, Tx^*)d(y^*, Ty^*)}{1 + d(Tx^*, Ty^*)}\}\\ &= d(x^*, y^*).\end{aligned} \tag{16}$$

Hence, we get

$$\psi(d(x^*, y^*)) \leq F(\psi(d(x^*, y^*)), \varphi(d(x^*, y^*))),$$

which implies $\psi(d(x^*, y^*)) = 0$ or $\varphi\left(d(x^*, y^*)\right) = 0$,so, $d(x^*, y^*) = 0$, that is, $x^* = y^*$. Hence T has a unique fixed point. ■

Corollary 2.3. *Let (X, d) be a complete GMS, $T : X \to X$ be a self mapping and $\alpha : X \times X \to [0, +\infty)$ a given function. Suppose that the following conditions are satisfied:*

(i) T is an α-admissible mapping and μ-subadmissible mapping;

(ii) $x, y \in X, h(\psi(d(Tx, Ty))) \leq \mu(x, y)\psi((M(x, y)) - \alpha(x, y)\varphi(M(x, y))$, where $\varphi, \psi \in \Phi$, h is a A-class function, and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \frac{d(x, Tx)d(y, Ty)}{1 + d(Tx, Ty)}\right\};$$

(iii) there exists $x_0 \in X$ such that

$$\alpha(\ x_0, T\ x_0) \geq 1, \alpha(\ x_0, T^2\ x_0) \geq 1 \text{ and } \mu(\ x_0, T\ x_0) \leq 1, \mu(\ x_0, T^2\ x_0) \leq 1;$$

(iv) either T is continuous, or X is α-regular and μ-subregular.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*. Further, if for all $x, y \in F(T)$, we have $\alpha(x,y) \geq 1, \mu(x,y) \leq 1$ then T has a unique fixed point in X.

Corollary 2.4. *Let (X,d) be a complete GMS, $T : X \to X$ be a self mapping and $\alpha : X \times X \to [0,+\infty)$ a given function. Suppose that the following conditions are satisfied:*

(i) T is an α-admissible mapping and μ-subadmissible mapping;

(ii) $x, y \in X, h(\psi(d(Tx,Ty))) \leq \frac{\mu(x,y)\psi((M(x,y))}{1+\alpha(x,y)\varphi(M(x,y))}$, where $\psi \in \Phi, \varphi \in \Phi_u$, h is a A-class function, and

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\};$$

(iii) there exists $x_0 \in X$ such that

$$\alpha(\ x_0, T\ x_0) \geq 1, \alpha(\ x_0, T^2\ x_0) \geq 1 \text{ and } \mu(\ x_0, T\ x_0) \leq 1, \mu(\ x_0, T^2\ x_0) \leq 1;$$

(iv) either T is continuous, or X is α-regular and μ-subregular.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*. Further, if for all $x, y \in F(T)$, we have $\alpha(x,y) \geq 1, \mu(x,y) \leq 1$ then T has a unique fixed point in X.

Definition 2.5. Let (X,d) be a generalized metric space and $\alpha, \mu : X \times X \to \mathbb{R}^+$. A mapping $T : X \to X$ is said to be T is an $(\alpha, \psi, \mathcal{F}, h)$-rational-1-1-upclass type-II contractive mapping if $(\alpha, \psi, \mathcal{F}, h)$-rational-1-1-upclass contractive mapping if for all $x, y \in X$ the following condition holds:

$$h(\psi(d(Tx,Ty))) \leq \mathcal{F}(\mu(x,y), \alpha(x,y), \psi((M(x,y)), \varphi(M(x,y)))), \tag{17}$$

where $\psi \in \Phi, \varphi \in \Phi_u$, $\mathcal{F}$ is a 1-1-upclass and h is a A-class function and

$$\begin{aligned} M(x,y) \quad = \quad & \max\{d(x,y), d(x,Tx), d(y,Ty), \\ & \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)+d(x,Ty)+d(y,Tx)}, \frac{d(x,Ty)d(Tx,y)}{1+d(x,Tx)+d(y,x)+d(y,Ty)}\}. \end{aligned}$$

For this class of mappings we state a similar existence and uniqueness theorem.

Theorem 2.6. *Let (X,d) be a complete generalized metric space, $T : X \to X$ be a self mapping, and $\alpha : X \times X \to \mathbb{R}$. Suppose that the following conditions are satisfied:*

(i) T is an α-admissible mapping,μ-subadmissible mapping;

(ii) T is an $(\alpha, \psi, \mathcal{F}, h)$-rational-1-1-upclass type-II contractive mapping;

(iii) there exists $x_0 \in X$ such that

$$\alpha(\ x_0, T\ x_0) \geq 1, \alpha(\ x_0, T^2\ x_0) \geq 1 \text{ and } \mu(\ x_0, T\ x_0) \leq 1, \mu(\ x_0, T^2\ x_0) \leq 1;$$

(iv) either T is continuous, or X is α-regular and μ-subregular.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*. Further, if for all $x, y \in F(T)$, we have $\alpha(x,y) \geq 1, \mu(x,y) \leq 1$, then T has a unique fixed point in X.

Proof. The proof follows similar steps as in Theorem 2.2 and is omitted for brevity. ■

Example 2.7. Let X be a finite set defined as $X = \{1,2,3,4\}$. Define $d : X \times X \to [0,+\infty)$ as:

$$\begin{aligned}
&d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0\\
&d(1,2) = d(2,1) = 3\\
&d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1\\
&d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = 4.
\end{aligned}$$

The function d is not a metric on X. Indeed, note that

$$3 = d(1,2) \geq d(1,3) + d(3,2) = 1 + 1 = 2,$$

that is, the triangle inequality is not satisfied. However, d is a generalized metric on X and moreover, (X,d) is a complete generalized metric space. Define $T : X \to X$ as

$$T1 = T2 = 1, T3 = 2, \quad T4 = 3,$$

$\mu(x,y)$ as $\mu(x,y) = 1$ $x, y \in \{1,2\}$,in other $\mu(x,y) = 10$,,and $\psi(t) = \frac{9t}{10}$. Then, for $x, y = 1, 2$ we have

$$d(Tx,Ty) = 0 \leq \mu(x,y)\psi(M(x,y)) = 0.$$

On the other hand, for $x = 1, 2$ and $y = 3$ we obtain

$$d(Tx,T3) = d(1,2) = 3$$

and

$$M(x,3) = \max\{d(x,3), d(x,T3), d(y,T3), \frac{d(x,Tx)d(y,Ty)}{1 + d(x,3)}, \frac{d(x,Tx)d(y,Ty)}{1 + d(Tx,T3)}\} = 3$$

and hence,

$$d(Tx,T3) = 3 \leq 10\frac{4}{2} = 20 = \mu(x,y)\psi(M(x,y)).$$

On the other hand, for $x = 1, 2$ and $y = 4$ we obtain

$$d(Tx,T4) = d(1,3) = 1$$

and

$$M(x,3) = \max\{d(x,4), d(x,T4), d(y,T4), \frac{d(x,Tx)d(y,Ty)}{1 + d(x,4)}, \frac{d(x,Tx)d(y,Ty)}{1 + d(Tx,T4)}\} = 4$$

and hence,

$$d(Tx,T4) = 1 \leq 10 \times 4 = 40 = \mu(x,y)\psi(M(x,y)).$$

For $x, y = 3, 4$, the contraction condition is obvious. Clearly, T satisfies the conditions of Theorem 2.2 and has a unique fixed point $x = 1$.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final version of manuscript.

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