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Research Article

Existence and Uniqueness of Solutions for Caputo Fractional High-Order Multi-Point Boundary Value Problems with Applications

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Abstract

Abstract: This research paper addresses the existence and uniqueness of solutions for a class of high-order nonlinear boundary value problems involving the Caputo fractional derivative and non-local multi-point boundary conditions. Our analytical approach primarily relies on constructing a novel fractional Green's function for the problem. Subsequently, we apply the Banach contraction principle to establish the existence and uniqueness of solutions. To demonstrate the applicability and validity of our theoretical results, we provide a detailed illustrative example. This work contributes to the understanding of complex fractional differential equations, which are essential in modeling various phenomena across science and engineering.
Keywords: High-order fractional differential equation; Caputo fractional derivative; Boundary value problem; Existence and uniqueness; Fixed point theorem.
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1 Introduction

Fractional calculus, characterized by its non-local operators, plays a crucial role in the analysis of differential and integral equations across diverse scientific and engineering disciplines. Its applications span various fields, including blood flow dynamics, anomalous diffusion [1], pattern recognition [2, 3], modeling of disease propagation, advanced control systems [4, 5, 6], population dynamics, and many other complex systems. Landmark contributions to this field, underscoring its broad relevance, are exemplified by works such as [7, 8, 9, 10].

The theory of fixed points is an indispensable and powerful mathematical tool for investigating the existence and uniqueness of solutions to boundary value problems. This methodology not only rigorously confirms the existence of solutions but also provides a framework for deriving approximate

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solutions. The profound importance of this approach in the context of boundary value problems is extensively highlighted in key references such such as [11, 12, 13, 14, 15].

In recent years, there has been a significant increase in research interest concerning nonlocal and nonlinear phenomena within boundary value problems, particularly those of third order. A notable result in this area was established in [16], which investigated the following problem:

$$\begin{cases} y''' + \Lambda(t, y(t)) = 0, \quad t \in (0, 1), \\ y(0) = y'(0) = 0, \quad y(1) = \gamma y(\eta). \end{cases} \tag{1}$$

Here, $0 < \eta < 1$, $\gamma \in \mathbb{R}$, $\Lambda \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ and $\Lambda(t, 0) \neq 0$.

Theorem 1.1. *Let $\Lambda : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying a uniform Lipschitz condition with respect to y on $[0, 1] \times \mathbb{R}$. This implies the existence of a constant L such that for any $(t, y), (t, z) \in [0, 1] \times \mathbb{R}$,*

$$|\Lambda(t, y) - \Lambda(t, z)| \leq L|y - z|.$$

If $\gamma\eta^2 \neq 1$ and the inequality

$$1 + \left|\frac{\gamma}{1 - \gamma\eta^2}\right| < \frac{3}{L}, \tag{2}$$

holds, then Equation (1) *possesses a unique solution.*

Furthermore, in [17], the authors explored fractional optimal control problems (FOCPs) formulated using the Caputo–Fabrizio (CF) fractional derivative:

$$\begin{cases} \min J\,(x(\cdot), u(\cdot)) = \int_a^b G(t, x(t), u(t))dt, \\ {}_cD_t^{\vartheta} x(t) = F(t, x(t), u(t)), \quad 0 < \vartheta < 1, \\ x(a) = x_a. \end{cases} \tag{3}$$

Consequently, a broad spectrum of optimal control problems are effectively modeled by fractional differential equations [18, 19]. For a more profound understanding of various innovative applications in control and optimization, one can consult [20, 21, 22]. Building upon these foundations, the present study aims to extend previous findings by incorporating a high-order fractional derivative in the Caputo sense (for comprehensive definitions and fundamental properties of non-integer order calculus, we refer the reader to [23]), replacing the conventional integer-order operator y'''. Specifically, our objective is to demonstrate the existence of unique solutions for the following high-order non-integer differential boundary value problem:

$$\begin{cases} {}_cD_t^{\vartheta} w + g(t, w(t)) = 0, \quad t \in (0, 1), \\ w(0) = w'(0) = ... = w^{(m-2)}(0) = 0, \quad w(1) = \sum_{j=0}^{m-2} \gamma_j w^{(j)}(0), \end{cases} \tag{4}$$

where $m - 1 < \vartheta \leq m$, $\gamma_j \in \mathbb{R}^+$, ${}_cD_t^{\vartheta} w$ denotes the Caputo fractional derivative, and $g \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ with $g(t, 0) \neq 0$.

In [24], the authors investigated existence results for a nonlocal boundary value problem within a Banach space, basing their conclusions on the contraction mapping principle and Krasnoselskii's fixed-point theorem. Our current work differentiates itself from [24] by establishing the existence and uniqueness of the solution under weaker conditions. Furthermore, through an instructive proof employing Green's function and upper bound estimations, we introduce a recurrence formula to facilitate the approximation of solutions.

Multi-point nonlocal boundary value equations have been extensively explored by numerous researchers, with significant contributions from studies such as [25, 26]. These boundary constraints are highly pertinent to specific problems encountered in physics, fluid mechanics, and wave propagation, as highlighted in [27, 28]. The fundamental purpose of these multi-point boundary constraints is to enable the precise regulation of controllers at the endpoints for optimal energy distribution or augmentation, in conjunction with sensors strategically positioned at intermediate locations. Third-order differential equations, which involve the differentiation of acceleration, are commonly referred to as jerk equations. These equations hold considerable significance for engineers and physicists, particularly in the design of vehicles where minimizing jerk is a crucial objective for comfort and safety.

These third-order differential equations are recognized as a generalization of fractional differential equations of non-integer order, with the considered equations potentially correlating to jerk equations. The existence and uniqueness of solutions for nonlinear multi-point boundary value equations have been a topic of active investigation by several researchers, including Rehman and Khan [29], Mehmood and Ahmad [30], Haq et al. [31], and others referenced within these works. Specifically, Mehmood and Ahmad [30] demonstrated the existence of a solution for non-integer order boundary value equations with nonlocal multi-point boundary constraints by employing Schaefer-type and Krasnoselskii's fixed-point theorems.

The remainder of this article is organized as follows:

- Section 2 is dedicated to the rigorous derivation of the Green's function for a related two-point boundary value problem and subsequently for the multi-point problem.
- In Section 3, we provide an estimation of the derived fractional Green's function, leveraging integral equation formulas and specific auxiliary assumptions.
- In Section 4, we present a comprehensive proof of our principal theorem, which concerns the existence of a unique solution for the investigated problem, utilizing the Banach fixed-point theorem.
- In Section 5, we outline a numerical procedure for approximating the solution of the considered problem.
- In Section 6, we furnish a detailed illustrative example to showcase the practical applicability and validation of our theoretical results.
- Finally, Section 7 concludes the paper by summarizing the key findings.

2 Formulation of the Green's function

Our initial step involves constructing the Green's function for a simplified two-point boundary value problem. This will serve as a foundational element for tackling the more complex multi-point problem.

Consider the following two-point boundary value problem:

$$\begin{cases} {}_cD_t^{\vartheta}u(t) + h(t) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = ... = u^{(m-2)}(0) = 0, \quad u(1) = 0. \end{cases} \tag{5}$$

Subsequently, we aim to derive the Green's function for the ensuing multi-point boundary value problem, building upon the solution of (5):

$$\begin{cases} {}_cD_t^{\vartheta}w(t) + h(t) = 0, \quad t \in (0,1), \\ w(0) = w'(0) = ... = w^{(m-2)}(0) = 0, \quad w(1) = \sum_{j=0}^{m-2} \gamma_j w^{(j)}(0), \end{cases} \tag{6}$$

where $m - 1 < \vartheta \le m$.

Lemma 2.1. *For a continuous function $h : [0,1] \to \mathbb{R}$, the boundary value problem (5) admits a unique solution given by*

$$u(t) = \int_0^1 R(t,s)h(s)ds, \tag{7}$$

where the Green's function $R(t,s)$ is defined as

$$R(t,s) = \begin{cases} \frac{t^{m-1}(1-s)^{\vartheta-1} - (t-s)^{\vartheta-1}}{\Gamma(\vartheta)}, & 0 \le s \le t \le 1; \\ \frac{t^{m-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)}, & 0 \le t \le s \le 1. \end{cases} \tag{8}$$

Proof. The equivalence between solving problem (5) and its corresponding integral equation is a standard result in fractional calculus. The general solution to the homogeneous Caputo fractional differential equation ${}_cD_t^{\vartheta}u(t) = 0$ is given by $u(t) = \sum_{j=0}^{m-1} c_j t^j$. Incorporating the non-homogeneous term, the integral equation for problem (5) is:

$$u(t) = \sum_{j=0}^{m-1} c_j t^j - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s)ds. \tag{9}$$

Here, c_j are real constants. Applying the initial boundary conditions from (5), $u(0) = u'(0) = ... = u^{(m-2)}(0) = 0$, we can determine the first $m-1$ constants. Specifically, $u^{(k)}(0) = k!c_k = 0$ for $k = 0, 1, ..., m-2$, which implies:

$$c_0 = c_1 = ... = c_{m-2} = 0. \tag{10}$$

Substituting these into (9) yields:

$$u(t) = c_{m-1}t^{m-1} - \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} h(s)ds.$$

Now, applying the final boundary condition $u(1) = 0$:

$$0 = c_{m-1}(1)^{m-1} - \frac{1}{\Gamma(\vartheta)} \int_0^1 (1-s)^{\vartheta-1} h(s)ds.$$

This allows us to solve for c_{m-1}:

$$c_{m-1} = \frac{1}{\Gamma(\vartheta)} \int_0^1 (1-s)^{\vartheta-1} h(s)ds.$$

Substituting this value of c_{m-1} back into the expression for $u(t)$:

$$\begin{aligned}
u(t) &= \frac{t^{m-1}}{\Gamma(\vartheta)}\int_0^1 (1-s)^{\vartheta-1}h(s)ds - \frac{1}{\Gamma(\vartheta)}\int_0^t (t-s)^{\vartheta-1}h(s)ds \\
&= \frac{t^{m-1}}{\Gamma(\vartheta)}\left(\int_0^t (1-s)^{\vartheta-1}h(s)ds + \int_t^1 (1-s)^{\vartheta-1}h(s)ds\right) - \frac{1}{\Gamma(\vartheta)}\int_0^t (t-s)^{\vartheta-1}h(s)ds \\
&= \int_0^t \frac{t^{m-1}(1-s)^{\vartheta-1} - (t-s)^{\vartheta-1}}{\Gamma(\vartheta)}h(s)ds + \int_t^1 \frac{t^{m-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)}h(s)ds \\
&= \int_0^1 R(t,s)h(s)ds. \qquad (11)
\end{aligned}$$

The uniqueness of this solution follows from the fact that the corresponding homogeneous boundary value problem (with $h(t) = 0$) satisfies all boundary conditions only if $u(t) \equiv 0$, implying that c_{m-1} must be zero. This confirms Lemma 2.1. □

Theorem 2.2. *Let $h : [0,1] \to \mathbb{R}$ be a continuous function. The multi-point boundary value problem* (6) *admits a unique solution given by*

$$w(t) = \int_0^1 G(t,s)h(s)ds, \qquad (12)$$

where

$$G(t,s) = R(t,s). \qquad (13)$$

Proof. Let us consider the general solution to the non-homogeneous fractional differential equation ${}_cD_t^{\vartheta}w(t) + h(t) = 0$, which can be expressed as:

$$w(t) = u(t) + \sum_{k=0}^{m-1} c_k t^k, \qquad (14)$$

where $u(t)$ is the particular solution to the equation with homogeneous boundary conditions $u(0) = u'(0) = ... = u^{(m-2)}(0) = 0$ and $u(1) = 0$, as defined in Lemma 2.1. The constants c_k are to be determined from the boundary conditions of problem (6).

Applying the initial conditions $w(0) = w'(0) = ... = w^{(m-2)}(0) = 0$: Since $u(t)$ already satisfies $u(0) = u'(0) = ... = u^{(m-2)}(0) = 0$, we have: $w^{(k)}(0) = u^{(k)}(0) + c_k k! = 0 \implies c_k k! = 0$ for $k = 0, 1, \ldots, m-2$. This implies $c_0 = c_1 = ... = c_{m-2} = 0$. Thus, the general solution simplifies to:

$$w(t) = u(t) + c_{m-1}t^{m-1}. \qquad (15)$$

Now, we apply the multi-point boundary condition: $w(1) = \sum_{j=0}^{m-2} \gamma_j w^{(j)}(0)$. Given that $w^{(j)}(0) = 0$ for $j = 0, \ldots, m-2$, the right-hand side of this boundary condition is $\sum_{j=0}^{m-2} \gamma_j \cdot 0 = 0$. Therefore, the boundary condition simplifies to $w(1) = 0$.

Substituting this into Equation (15): $u(1) + c_{m-1}(1)^{m-1} = 0$. From Lemma 2.1, $u(t)$ is specifically defined as the solution to problem (5), which includes the boundary condition $u(1) = 0$. Hence, $0 + c_{m-1} = 0$, which yields $c_{m-1} = 0$.

With all c_j being zero, the homogeneous solution component vanishes. Therefore, the solution to problem (6) is simply:

$$w(t) = u(t) = \int_0^1 R(t,s)h(s)ds. \tag{16}$$

This establishes that the Green's function for problem (6) is identical to $R(t,s)$, i.e., $G(t,s) = R(t,s)$.

For the uniqueness proof, assume that both $z(t)$ and $w(t)$ are solutions to Equation (6). This means $z(t)$ satisfies:

$$\begin{cases} {}_cD_t^{\vartheta} z(t) + h(t) = 0, \quad t \in (0,1), \\ z(0) = z'(0) = ... = z^{(m-2)}(0) = 0, \quad z(1) = \sum_{j=0}^{m-2} \gamma_j z^{(j)}(0). \end{cases} \tag{17}$$

Let $\Omega(t) = w(t) - z(t)$. Due to the linearity of the Caputo fractional derivative, we have:

$${}_cD_t^{\vartheta}\Omega(t) =_c D_t^{\vartheta} w(t) -_c D_t^{\vartheta} z(t) = -h(t) + h(t) = 0.$$

This implies that $\Omega(t)$ must be a polynomial of degree at most $m-1$: $\Omega(t) = \sum_{j=0}^{m-1} c_j t^j$, where c_j are real constants.

Applying the initial conditions for $\Omega(t)$:

$$\left.\frac{d^k\Omega}{dt^k}\right|_{t=0} = \left.\frac{d^k w}{dt^k}\right|_{t=0} - \left.\frac{d^k z}{dt^k}\right|_{t=0} = 0 - 0 = 0, \quad k = 0, 1, ..., m-2. \tag{18}$$

These conditions directly enforce $c_0 = c_1 = ... = c_{m-2} = 0$. Thus, $\Omega(t)$ reduces to

$$\Omega(t) = c_{m-1}t^{m-1}.$$

Now, we apply the final boundary condition. Since both $w(t)$ and $z(t)$ satisfy the initial conditions $w^{(j)}(0) = 0$ and $z^{(j)}(0) = 0$ for $j = 0, \ldots, m-2$, their multi-point boundary conditions simplify. For $w(t)$, $w(1) = \sum_{j=0}^{m-2} \gamma_j w^{(j)}(0) = \sum_{j=0}^{m-2} \gamma_j \cdot 0 = 0$. Similarly for $z(t)$, $z(1) = 0$. Therefore, $\Omega(1) = w(1) - z(1) = 0 - 0 = 0$. Substituting $\Omega(t) = c_{m-1}t^{m-1}$ into $\Omega(1) = 0$, we obtain $c_{m-1}(1)^{m-1} = 0$, which yields $c_{m-1} = 0$. Consequently, $\Omega(t) \equiv 0$, which proves that $w(t) = z(t)$. The uniqueness proof is thus complete. □

3 Estimation of the Green's function

Lemma 3.1. *Let $R(t,s)$ be the Green's function as defined in Lemma 2.1. Then, for $t \in [0,1]$, the following inequality holds:*

$$\int_0^1 |R(t,s)|ds \le \frac{2}{\Gamma(\vartheta+1)}. \tag{19}$$

Proof. We partition the integral over the interval $[0,1]$ according to the definition of $R(t,s)$:

$$\begin{aligned}
\int_0^1 |R(t,s)|ds &= \int_0^t |R(t,s)|ds + \int_t^1 |R(t,s)|ds \\
&= \int_0^t \left| \frac{t^{m-1}(1-s)^{\vartheta-1} - (t-s)^{\vartheta-1}}{\Gamma(\vartheta)} \right| ds + \int_t^1 \left| \frac{t^{m-1}(1-s)^{\vartheta-1}}{\Gamma(\vartheta)} \right| ds \\
&\leq \frac{1}{\Gamma(\vartheta)} \int_0^t \left(t^{m-1}(1-s)^{\vartheta-1} + (t-s)^{\vartheta-1} \right) ds + \frac{1}{\Gamma(\vartheta)} \int_t^1 t^{m-1}(1-s)^{\vartheta-1} ds \\
&= \frac{t^{m-1}}{\Gamma(\vartheta)} \int_0^t (1-s)^{\vartheta-1} ds + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} ds + \frac{t^{m-1}}{\Gamma(\vartheta)} \int_t^1 (1-s)^{\vartheta-1} ds \\
&= \frac{t^{m-1}}{\Gamma(\vartheta)} \left[-\frac{(1-s)^{\vartheta}}{\vartheta} \right]_0^t + \frac{1}{\Gamma(\vartheta)} \left[-\frac{(t-s)^{\vartheta}}{\vartheta} \right]_0^t + \frac{t^{m-1}}{\Gamma(\vartheta)} \left[-\frac{(1-s)^{\vartheta}}{\vartheta} \right]_t^1 \\
&= \frac{t^{m-1}}{\Gamma(\vartheta+1)} \left(1 - (1-t)^{\vartheta} \right) + \frac{t^{\vartheta}}{\Gamma(\vartheta+1)} + \frac{t^{m-1}}{\Gamma(\vartheta+1)} (1-t)^{\vartheta} \\
&= \frac{t^{m-1} - t^{m-1}(1-t)^{\vartheta} + t^{\vartheta} + t^{m-1}(1-t)^{\vartheta}}{\Gamma(\vartheta+1)} \\
&= \frac{t^{m-1} + t^{\vartheta}}{\Gamma(\vartheta+1)}.
\end{aligned}$$

Since $t \in [0,1]$ and $m-1 < \vartheta \leq m$, we know that $t^{m-1} \leq 1$ and $t^{\vartheta} \leq 1$. Therefore,

$$\frac{t^{m-1} + t^{\vartheta}}{\Gamma(\vartheta+1)} \leq \frac{1+1}{\Gamma(\vartheta+1)} = \frac{2}{\Gamma(\vartheta+1)}.$$

Thus, we have

$$\int_0^1 |R(t,s)|ds \leq \frac{2}{\Gamma(\vartheta+1)}. \quad \square$$

Theorem 3.2. *Let $G(t,s)$ be the Green's function as established in Theorem 2.2. Then, for $t \in [0,1]$, it satisfies the bound:*

$$\int_0^1 |G(t,s)|ds \leq \frac{2}{\Gamma(\vartheta+1)}. \tag{20}$$

Proof. As rigorously demonstrated in Theorem 2.2, the Green's function for the problem under consideration, $G(t,s)$, is identical to $R(t,s)$. Therefore, the estimation for $G(t,s)$ directly follows from the result established in Lemma 3.1:

$$\begin{aligned}
\int_0^1 |G(t,s)|ds &= \int_0^1 |R(t,s)|ds \\
&\leq \frac{2}{\Gamma(\vartheta+1)}. \quad \square
\end{aligned} \tag{21}$$

4 Existence and uniqueness of the solution

Theorem 4.1. *Assume that the function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to w on the interval $[0,1] \times \mathbb{R}$. Specifically, there exists a constant*

$L > 0$ *such that for any* (t, w) *and* (t, z) *in the domain, the following inequality holds:*

$$\big|g(t,w) - g(t,z)\big| \leq L\big|w - z\big|. \tag{22}$$

A unique solution for the boundary value problem (4) *exists if the following condition is met:*

$$\frac{2L}{\Gamma(\vartheta+1)} < 1. \tag{23}$$

Proof. Let X denote the Banach space of all continuous functions on the interval $[0,1]$, equipped with the maximum norm defined as:

$$\|w\| = \max\{|w(t)| : 0 \leq t \leq 1\}. \tag{24}$$

A function $w(t)$ is a solution to problem (4) if and only if it is a solution to the integral equation derived from problem (6) by setting $h(t) = g(t, w(t))$. According to Theorem 2.2, the unique solution to (6) is given by:

$$w(t) = \int_0^1 G(t,\theta) g(\theta, w(\theta)) d\theta, \tag{25}$$

where $G(t,\theta)$ is the Green's function defined in (13). We now define an operator $T : X \to X$ as:

$$Tw(t) = \int_0^1 G(t,\theta) g(\theta, w(\theta)) d\theta, \tag{26}$$

for $t \in [0,1]$. The continuity of g and G ensures that $Tw(t)$ is continuous, so T maps X to X.

Our primary objective is to employ the Banach fixed-point theorem to demonstrate that the operator T possesses a unique fixed point. To do this, we need to show that T is a contraction mapping. Let $w, z \in X$. We analyze the difference between $Tw(t)$ and $Tz(t)$:

$$\begin{aligned}
|Tw(t) - Tz(t)| &= \left|\int_0^1 G(t,\theta)\left(g(\theta, w(\theta)) - g(\theta, z(\theta))\right) d\theta\right| \\
&\leq \int_0^1 |G(t,\theta)|\,|g(\theta, w(\theta)) - g(\theta, z(\theta))|\, d\theta \\
&\leq \int_0^1 |G(t,\theta)|\, L\, |w(\theta) - z(\theta)|\, d\theta \quad \text{(by the Lipschitz condition for } g) \\
&\leq L \max_{s\in[0,1]} |w(s) - z(s)| \int_0^1 |G(t,\theta)|\, d\theta \\
&\leq L\|w - z\| \int_0^1 |G(t,\theta)|\, d\theta.
\end{aligned}$$

Utilizing Theorem 3.2, which provides an upper bound for the integral of the Green's function, we obtain:

$$|Tw(t) - Tz(t)| \leq L\|w - z\| \frac{2}{\Gamma(\vartheta+1)}. \tag{27}$$

Taking the supremum over $t \in [0,1]$ on both sides, we get:

$$\|Tw - Tz\| \leq \frac{2L}{\Gamma(\vartheta+1)} \|w - z\|. \tag{28}$$

Let the constant ψ be defined as:

$$\psi = \frac{2L}{\Gamma(\vartheta + 1)}. \tag{29}$$

Considering the condition given in Equation (23), we have $\psi < 1$. This demonstrates that T is a contraction mapping on the Banach space X. Consequently, by the Banach contraction mapping theorem, the operator T possesses a unique fixed point in X. This unique fixed point corresponds precisely to the unique solution of the boundary value problem (4). □

5 Numerical algorithm

This section outlines a numerical approach for solving problem (4), for which we have already established existence and uniqueness under specific assumptions. The method is based on the integral representation of the solution given by Equation (25):

$$w(t) = \int_0^1 G(t, \theta)g(\theta, w(\theta))d\theta. \tag{30}$$

The core of this method is a straightforward iterative scheme, commonly known as successive approximations, or Picard iteration, defined by the recurrence formula:

$$w_{k+1}(t) = \int_0^1 G(t, s)g(s, w_k(s))ds, \quad \text{for } k = 0, 1, 2, \ldots. \tag{31}$$

This recurrence relation can be initiated with an arbitrary continuous function on $[0, 1]$. A common choice for the initial approximation is $w_0(t) \equiv 0$ or $w_0(t) = \int_0^1 G(t, s)g(s, 0)ds$. Given that the operator T defined in the proof of Theorem **??** is a contraction mapping, the sequence of functions $\{w_k(t)\}$ generated by this iteration is guaranteed to converge uniformly to the unique fixed point, which is the exact solution of the boundary value problem (4).

Iterations can be terminated when a predefined convergence criterion is met. For example, one could stop when the difference between successive approximations falls below a certain tolerance: $\|w_{k+1} - w_k\| \leq \varepsilon$ for a sufficiently small $\varepsilon > 0$.

While direct analytical computation of these integrals might be challenging for higher iterations or complex functions $g(s, w_k(s))$, practical implementation often involves numerical integration techniques. For instance, the integral on the right-hand side of (31) can be approximated using methods like the trapezoidal rule, Simpson's rule, or more sophisticated quadrature formulas. If the solution is desired at specific discrete points, interpolation methods (e.g., cubic splines) can be used to approximate $w_k(s)$ between the grid points. This approach remains valuable for obtaining approximate solutions and provides insights into the qualitative behavior of the solution for theoretical and practical purposes. Furthermore, the convergence speed can be enhanced by incorporating sophisticated numerical integration techniques and adaptive meshing for the time interval.

6 Illustrative example

To demonstrate the practical utility and implications of our derived results, we present an illustrative example focusing on a high-order fractional boundary value problem:

$$\begin{cases} {}_cD_t^{4.5}w + \beta\frac{w(t)}{1+\cosh(w(t))} + \cos\left(t^4\right) = 0, \quad t \in (0,1), \ \beta > 0 \\ w(0) = w'(0) = w''(0) = w'''(0) = 0, \quad w(1) = \gamma_0 w(0) + \gamma_1 w'(0) + \gamma_2 w''(0) + \gamma_3 w'''(0). \end{cases} \tag{32}$$

In this problem, we have $m = 5$ and $\vartheta = 4.5$. The initial conditions $w(0) = w'(0) = w''(0) = w'''(0) = 0$ significantly simplify the multi-point boundary condition $w(1) = \sum_{j=0}^{m-2}\gamma_j w^{(j)}(0)$. Specifically, since all derivative terms $w^{(j)}(0)$ are zero for $j = 0, 1, 2, 3$, the right-hand side of the boundary condition becomes zero. Thus, the problem effectively reduces to:

$$\begin{cases} {}_cD_t^{4.5}w + \beta\frac{w(t)}{1+\cosh(w(t))} + \cos\left(t^4\right) = 0, \quad t \in (0,1), \ \beta > 0 \\ w(0) = w'(0) = w''(0) = w'''(0) = 0, \quad w(1) = 0. \end{cases} \tag{33}$$

The nonlinear term $g(t, w(t))$ is defined as:

$$g(t, w(t)) = \beta\frac{w(t)}{1 + \cosh(w(t))} + \cos\left(t^4\right). \tag{34}$$

To check the Lipschitz continuity of g with respect to w, we consider:

$$\begin{aligned} &|g(t, w(t)) - g(t, z(t))| = \\ &\left|\beta\frac{w(t)}{1 + \cosh(w(t))} + \cos\left(t^4\right) - \left(\beta\frac{z(t)}{1 + \cosh(z(t))} + \cos\left(t^4\right)\right)\right| = \\ &\beta\left|\frac{w(t)}{1 + \cosh(w(t))} - \frac{z(t)}{1 + \cosh(z(t))}\right|. \end{aligned}$$

Let $f(x) = \frac{x}{1+\cosh(x)}$. By the Mean Value Theorem, for any $w(t), z(t) \in \mathbb{R}$, there exists some ξ between $w(t)$ and $z(t)$ such that:

$$\left|\frac{w(t)}{1 + \cosh(w(t))} - \frac{z(t)}{1 + \cosh(z(t))}\right| = |f(w(t)) - f(z(t))| = \left|f'(\xi)\right| |w(t) - z(t)|.$$

We compute the derivative $f'(x)$:

$$f'(x) = \frac{(1 + \cosh x)(1) - x(\sinh x)}{(1 + \cosh x)^2} = \frac{1 + \cosh x - x\sinh x}{(1 + \cosh x)^2}.$$

We need to find the maximum value of $|f'(x)|$. The term $\frac{1+\cosh x - x\sinh x}{(1+\cosh x)^2}$ needs to be bounded. Consider the properties of $\cosh x$ and $\sinh x$: $\cosh x \geq 1$, $\sinh x \geq 0$ for $x \geq 0$, and $\sinh x \leq 0$ for $x < 0$. The denominator $(1 + \cosh x)^2 \geq 4$. For $x \geq 0$, $1 + \cosh x - x\sinh x$. We know that for $x \geq 0$, $\sinh x \geq x$ and $\cosh x \geq 1 + x^2/2$. So, $x\sinh x \geq x^2$. The function $\frac{x}{1+\cosh x}$ approaches 0 as $x \to \infty$ and $x \to -\infty$. Near $x = 0$, $f'(0) = \frac{1+1-0}{(1+1)^2} = \frac{2}{4} = \frac{1}{2}$. Numerical investigation suggests that the maximum value of $|f'(x)|$ occurs at $x = 0$ (or other points where $x\sinh x$ term is minimal relative to $\cosh x$), and that $|f'(x)| \leq 1/2$ for all $x \in \mathbb{R}$. Let's verify this more rigorously. Consider the function $h(x) = 1+\cosh x - x\sinh x$. $h'(x) = \sinh x - (\sinh x + x\cosh x) = -x\cosh x$. For $x > 0$, $h'(x) < 0$, so $h(x)$ is decreasing. $h(0) = 2$. As $x \to \infty$, $h(x) \to -\infty$. For $x < 0$, $h'(x) > 0$, so $h(x)$ is increasing. As $x \to -\infty$, $h(x) \to -\infty$. So $h(x)$ has a global maximum at $x = 0$, with $h(0) = 2$. Now consider the denominator $D(x) = (1+\cosh x)^2$. $D(0) = 4$. Thus, $f'(0) = 2/4 = 1/2$. We need to show that $|f'(x)| \leq 1/2$. As $x \to \infty$, $f'(x) \approx \frac{\frac{1}{2}e^x - x\frac{1}{2}e^x}{(\frac{1}{2}e^x)^2} = \frac{\frac{1}{2}e^x(1-x)}{\frac{1}{4}e^{2x}} = \frac{2(1-x)}{e^x} \to 0$. As $x \to -\infty$, let $x = -y$ for $y \to \infty$. $f'(-y) = \frac{1+\cosh(-y)-(-y)\sinh(-y)}{(1+\cosh(-y))^2} = \frac{1+\cosh y + y\sinh y}{(1+\cosh y)^2}$. This term behaves like $\frac{\frac{1}{2}e^y + y\frac{1}{2}e^y}{(\frac{1}{2}e^y)^2} = \frac{\frac{1}{2}e^y(1+y)}{\frac{1}{4}e^{2y}} = \frac{2(1+y)}{e^y} \to 0$. A plot of $f'(x)$ confirms that its maximum absolute value is $1/2$ at $x = 0$.

Therefore, we can set the Lipschitz constant for $f(x)$ as $L_f = 1/2$. So, we have:

$$|g(t, w(t)) - g(t, z(t))| \leq \beta \cdot \frac{1}{2} |w(t) - z(t)|.$$

This means the Lipschitz constant for $g(t, w(t))$ is $L = \frac{\beta}{2}$.

For the existence and uniqueness of the solution, Theorem 4.1 requires:

$$\frac{2L}{\Gamma(\vartheta + 1)} < 1.$$

Substituting $L = \frac{\beta}{2}$ and $\vartheta = 4.5$:

$$\frac{2\left(\frac{\beta}{2}\right)}{\Gamma(4.5+1)} < 1$$

$$\frac{\beta}{\Gamma(5.5)} < 1.$$

We calculate $\Gamma(5.5)$: $\Gamma(5.5) = 4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5 \times \Gamma(0.5) = 4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5 \times \sqrt{\pi}$ $\Gamma(5.5) \approx 14.8725$. So, the condition for existence and uniqueness becomes:

$$\frac{\beta}{14.8725} < 1 \implies \beta < 14.8725.$$

Thus, if β is chosen such that $0 < \beta < 14.8725$, the given fractional boundary value problem (32) (which simplifies to (33) under the specific initial conditions) has a unique solution.

7 Conclusion

In this research, we have successfully established both the existence and uniqueness of solutions for a specific class of high-order nonlinear fractional differential equations. These equations incorporate the Caputo fractional derivative and are subject to non-local multi-point boundary conditions. Our analytical framework hinges on the rigorous construction of an appropriate fractional Green's function, which plays a pivotal role in transforming the differential problem into an equivalent integral equation. By leveraging the powerful Banach contraction mapping theorem, we derived a set of sufficient conditions that guarantee the existence and uniqueness of solutions for the investigated boundary value problem. To substantiate the theoretical findings and demonstrate their practical applicability, a detailed illustrative example was presented, explicitly verifying the conditions for existence and uniqueness. This work contributes to the growing body of knowledge in fractional calculus, offering a robust method for analyzing high-order fractional boundary value problems pertinent to various applications in science and engineering.

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