Zagros J. Appl. Math. & Data. Anal.

Available online at https://zjamda.abru.ac.ir/
ISSN 0000-0000
Vol. 1, No. 2, 2025, 49–70
*Research Article*

# A Hybrid Double Sumudu Transform and Optimized Multistage ADM Framework for Solving Nonlinear Time-Fractional PDEs with Mixed Boundary Conditions

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**Abstract**

**Abstract:** This paper develops a rigorous hybrid analytical–numerical framework that combines the Double Sumudu Transform (DST) with a parameter-optimized multistage Adomian Decomposition Method (ADM) for the solution of nonlinear time-fractional partial differential equations (PDEs) subject to mixed nonlinear boundary conditions. In this approach, the integral-transform properties of the DST are employed to reduce the complexity of fractional operators, while the multistage ADM, enhanced by metaheuristic parameter tuning, improves the convergence rate and reduces computational overhead. A theoretical foundation is provided by establishing error bounds and stability criteria in suitable Banach spaces, with explicit assumptions on the nonlinear operators. The accuracy and efficiency of the method are validated through benchmark problems, including the time-fractional heat equation, fractional Klein–Gordon equation, and nonlinear reaction–diffusion models with Robin- and cubic-type boundary conditions. Numerical comparisons against classical ADM, recent transform-based schemes, and numerical methods such as RBF-FD and spectral techniques demonstrate improved convergence, lower root-mean-square errors, and competitive computational cost. The results confirm that the proposed hybrid framework constitutes a mathematically consistent and practically efficient tool for solving complex nonlinear fractional PDEs, and it provides a viable alternative to standard transform or decomposition methods.

**Keywords:** Double Sumudu Transform, Adomian Decomposition Method, Time-Fractional PDEs, Parameter Optimization, Stability Analysis.
**2020 Mathematics Subject Classification:** 65M60; 65N30.

## 1 Introduction

Nonlinear partial differential equations (PDEs) play a fundamental role in modeling diverse physical and engineering phenomena, ranging from anomalous diffusion and viscoelasticity to wave propa-

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gation in heterogeneous media [1, 2]. When fractional derivatives are incorporated, such models capture memory and hereditary effects that classical integer-order PDEs fail to describe adequately [3, 4, 5]. The theory of fractional operators and nonlocal dynamics has thus become a cornerstone of modern applied mathematics [1, 6].

Despite this potential, the analytical and numerical treatment of nonlinear time-fractional PDEs with complex nonlinearities and mixed boundary conditions remains highly challenging [7, 8, 9, 10, 11, 12]. Classical numerical techniques—including finite difference, finite element, alternating direction implicit (ADI), and spectral schemes—achieve high accuracy in certain regimes but often suffer from stability issues, stiffness, or high computational cost in multidimensional problems [7, 8, 9, 10, 11, 13, 14, 15]. To overcome mesh dependency and domain irregularities, meshless methods based on radial basis functions (RBFs) and local weak formulations have been proposed for time-fractional diffusion, wave-diffusion, and telegraph equations [16, 17, 18, 19, 20, 21, 22], offering flexibility and accuracy for higher-dimensional problems.

In parallel, semi-analytical decomposition techniques have been developed as alternatives to purely numerical methods. The Adomian Decomposition Method (ADM) [23, 24, 25], the Variational Iteration Method (VIM) [26], and the Homotopy Analysis Method (HAM) [6] are notable representatives. These approaches yield approximate series solutions without linearization or small-parameter assumptions. However, classical ADM often converges slowly or lacks stability for problems with strong nonlinearities and nonstandard boundary conditions [27, 28]. Recent modifications—including optimized and hybrid decomposition schemes—have addressed these limitations, improving convergence and robustness [29, 30, 31, 32, 33]. For instance, parameter-optimized multistage ADM formulations have been shown to reduce truncation errors and accelerate convergence in nonlinear fractional PDEs [29, 30, 31].

Hybrid transform-based frameworks constitute another active research direction. Transform operators such as Laplace and Sumudu are widely employed to simplify fractional derivatives, and recent efforts have focused on generalized forms such as the Double ARA–Sumudu Transform (DAST) and the Double Sumudu Transform (DST) [34, 35]. These have been effectively combined with ADM to address nonlinear boundary value problems [34, 35, 31]. Notably, Yang-type transforms and hybrid decomposition-transform strategies have further enhanced convergence properties for nonlinear fractional operators [31, 33, 36]. More recently, works have considered multi-continuum and heterogeneous media, highlighting the importance of fractional PDE solvers in realistic applications [37, 38, 22].

Alongside these developments, rigorous theoretical analyses have emerged. Stability and error estimates for nonlocal operators are now established in various contexts, including Robin boundary conditions and convection–reaction–diffusion equations [3, 39, 24]. Studies of well-posedness for fractional diffusion and wave-type equations with memory, time-dependent coefficients, and nonlinear damping have also advanced the mathematical foundations [4, 40, 5]. High-order compact and spectral schemes further reinforce accuracy guarantees [13, 14, 15], and uniqueness and identifiability results for inverse problems add theoretical depth [12, 38].

Another emerging theme is the integration of decomposition frameworks with optimization paradigms. Metaheuristic and nature-inspired optimization algorithms—such as particle swarm optimization (PSO) [41, 42], evolutionary strategies [43], and swarm intelligence approaches [44, 45]—have been adopted to automatically tune decomposition parameters [20, 30]. Such hybridization accelerates convergence, expands the effective radius of analyticity, and reduces computational overhead, thereby bridging semi-analytical and numerical domains.

Despite these significant advances, current approaches often address only subsets of the difficulties inherent to nonlinear fractional PDEs. ADM-based hybrids improve convergence but are not always efficient for high-order or mixed boundary conditions; purely numerical schemes may lack scalability in nonlinear fractional operators; and transform-based methods frequently trade generality for tractability. A unified framework that integrates the DST, parameter-optimized multistage ADM, and rigorous stability analysis for nonlinear time-fractional PDEs with mixed nonlinear boundary conditions remains largely unexplored.

**Objectives and Contributions.** This paper fills this gap by developing a hybrid analytical–numerical framework that:

- integrates the Double Sumudu Transform (DST) with a parameter-optimized multistage ADM to address nonlinear time-fractional PDEs with mixed nonlinear boundary conditions [34, 35, 31];
- employs metaheuristic optimization techniques inspired by swarm intelligence to fine-tune decomposition parameters, leveraging advances in PSO and evolutionary computing [41, 44, 45, 43, 42];
- establishing error bounds and stability criteria in suitable Banach spaces (see Section 4), building upon recent theoretical contributions [3, 4, 40, 5];
- validates the proposed framework through benchmark problems, including the time-fractional heat, Klein–Gordon, and nonlinear reaction–diffusion equations [7, 9, 8, 21];
- provides comparative results against state-of-the-art numerical and hybrid methods, including compact finite difference, spectral, and RBF-based approaches [13, 14, 15, 16, 18].

**Organization of the Paper.** The remainder of the paper is structured as follows. Section 2 introduces the mathematical formulation and preliminaries. Section 3 describes the proposed hybrid methodology. Section 4 establishes the theoretical stability and convergence analysis. Section 5 presents numerical experiments and comparative evaluations. Section 6 discusses implications, limitations, and future research directions. Finally, Section 7 concludes the paper.

## 2 Problem Formulation

We consider a general class of nonlinear time-fractional partial differential equations (PDEs) of the form

$$\mathcal{D}_t^{\alpha} u(x,t) = \mathcal{L}_x[u(x,t)] + \mathcal{N}[u(x,t)] + f(x,t), \quad (x,t) \in (0,L) \times (0,T], \quad 0 < \alpha \leq 1, \tag{1}$$

subject to mixed nonlinear boundary conditions

$$\mathcal{B}_1[u] := G_1(u(0,t), u_x(0,t)) = g_1(t), \quad t \in (0,T], \tag{2}$$

$$\mathcal{B}_2[u] := G_2(u(L,t), u_x(L,t)) = g_2(t), \quad t \in (0,T], \tag{3}$$

and the initial condition

$$u(x,0) = \phi(x), \quad x \in [0,L]. \tag{4}$$

Here, $\mathcal{D}_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha$ [1], defined for a sufficiently smooth function $u$ as

$$\mathcal{D}_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} (t-\tau)^{-\alpha}\, d\tau, \quad 0<\alpha<1, \tag{5}$$

with $\mathcal{D}_t^1 u = \frac{\partial u}{\partial t}$ when $\alpha = 1$. The operator $\mathcal{L}_x$ denotes a linear spatial differential operator (e.g., diffusion, convection–diffusion, or wave operator), while $\mathcal{N}$ represents a general nonlinear operator, possibly polynomial or non-polynomial in $u$ and its derivatives. The term $f(x,t)$ is an external source. The boundary operators $G_1$ and $G_2$ in (2)–(3) may encode linear, Robin-type, or fully nonlinear conditions (e.g., cubic boundary feedback), which will be specified in the numerical experiments.

## 2.1 Functional Setting and Mild Solution

We briefly outline the functional framework and the concept of a mild solution for the problem (1)–(4). The Caputo fractional derivative $\mathcal{D}_t^\alpha$ defined in (5) can be expressed in terms of the Riemann-Liouville fractional integral $I_t^{1-\alpha}$ for $0<\alpha<1$:

$$\mathcal{D}_t^\alpha u(t) = I_t^{1-\alpha}\left(\frac{\partial u}{\partial t}\right)(t), \quad \text{where} \quad I_t^\beta v(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} v(s)\, ds.$$

The linear operator $\mathcal{L}_x$, equipped with the boundary conditions (2)–(3), is assumed to generate a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on $L^2(0,L)$. The domain $D(\mathcal{L}_x)$ is typically a subspace of $H^2(0,L)$ incorporating the boundary conditions, e.g.,

$$D(\mathcal{L}_x) = \left\{u \in H^2(0,L) : G_1(u(0), u_x(0)) = 0,\ G_2(u(L), u_x(L)) = 0\right\}.$$

A function $u \in C([0,T]; L^2(0,L))$ is called a mild solution of (1)–(4) if it satisfies the following integral equation formulated using the semigroup and the fractional integral:

$$u(t) = S(t)\phi + \int_0^t (t-s)^{\alpha-1} S(t-s)\left[\mathcal{N}[u(s)] + f(s)\right] ds, \tag{6}$$

where the semigroup $S(t)$ incorporates the action of the linear operator $\mathcal{L}_x$ and the boundary conditions. The well-posedness of this mild solution under the assumptions stated in Section 2.2 follows from standard fixed-point arguments and fractional semigroup theory [2, 4, 5].

## 2.2 Theoretical Assumptions

To ensure the well-posedness of (1)–(4), we impose the following assumptions, consistent with the standard theory of fractional evolution equations [1, 2]:

- **Fractional order.** The order of the derivative satisfies $0<\alpha\leq 1$. The Caputo derivative (5) is well-defined for functions $u$ such that $\frac{\partial u}{\partial \tau} \in L^1(0,T; L^2(0,L))$, or equivalently, $u \in AC([0,T]; L^2(0,L))$, the space of absolutely continuous functions on $[0,T]$ with values in $L^2(0,L)$.

- **Solution space.** The solution $u(x,t)$ is sought in the Banach space

$$X := C\big([0,T]; H^2(0,L)\big) \cap AC\big([0,T]; L^2(0,L)\big),$$

where $AC$ denotes the space of absolutely continuous functions, ensuring the Caputo derivative is well-defined.

- **Compatibility condition.** The initial condition $\phi(x)$ and boundary data $g_1(t), g_2(t)$ are assumed to be compatible at $t = 0$. That is:

$$G_1(\phi(0), \phi'(0)) = g_1(0), \quad G_2(\phi(L), \phi'(L)) = g_2(0).$$

This ensures the existence of a smooth solution and is crucial for the consistency of numerical methods.

- **Nonlinear operator.** The nonlinear operator $\mathcal{N} : X \to L^2(0,L)$ is assumed to be Lipschitz continuous on bounded subsets of $X$. Specifically, for any $u, v$ in a ball $B_R \subset X$, there exists a constant $L_{\mathcal{N}}(R) > 0$ such that

$$\|\mathcal{N}[u] - \mathcal{N}[v]\|_{L^2} \le L_{\mathcal{N}} \|u - v\|_{L^2}.$$

If $\mathcal{N}$ involves spatial derivatives (e.g., $uu_x$), the Lipschitz condition must be interpreted in the appropriate higher-order norm (e.g., $H^1$).

- **Linear operator.** The operator $\mathcal{L}_x$ is linear and densely defined on $L^2(0,L)$ with domain $D(\mathcal{L}_x) = H^2(0,L) \cap H_0^1(0,L)$ or another suitable domain incorporating the boundary conditions (2)–(3). It is assumed to be the generator of a strongly continuous semigroup $\{S(t)\}_{t\ge 0}$ on $L^2(0,L)$. For instance, the Laplace operator $\mathcal{L}_x = \partial_{xx}$ with Robin boundary conditions $au + bu_x = g$ satisfies this assumption [2].

- **Source term.** The forcing term $f(x,t)$ is continuous and bounded on $[0,L] \times [0,T]$, i.e., $\|f\|_{L^\infty([0,L]\times[0,T])} < \infty$. More generally, $f \in L^2(0,T; L^2(0,L))$ is sufficient for the well-posedness theory.

Under these assumptions, equation (1) admits a unique mild solution [2, 4, 5] in the sense of fractional semigroup theory. The subsequent sections develop a hybrid numerical–analytical strategy, based on the Double Sumudu Transform (DST) and an optimized multistage ADM, for constructing accurate approximations of this solution.

# 3 Hybrid DST–ADM Algorithm

This section presents the detailed procedure of the proposed hybrid method that combines the Double Sumudu Transform Technique (DST) with an optimized multistage Adomian Decomposition Method (ADM). The goal is to exploit the algebraic simplification properties of the DST and the convergence-enhancing capabilities of the stabilized ADM to construct accurate approximations for nonlinear time-fractional PDEs. The efficacy of the double Sumudu transform (DST) in handling fractional operators is well-documented [34, 35]. Its primary advantage lies in its ability to reduce

Caputo fractional derivatives to algebraic expressions in the transform domain, thereby simplifying the structure of the problem. The fundamental mathematical properties of integral transforms for fractional derivatives, including the necessary convergence conditions, are established in [1]. In this work, we leverage this property to transform the nonlinear time-fractional PDE into a more tractable form. The subsequent solution process employs an optimized multistage Adomian decomposition, where the transformed equation is solved iteratively.

### 3.1 Algorithm Steps

Consider the nonlinear time-fractional PDE of the form

$$\mathcal{D}_t^\alpha u(x,t) = \mathcal{L}[u(x,t)] + \mathcal{N}[u(x,t)] + f(x,t), \quad (x,t) \in (0,L) \times (0,T], \quad 0 < \alpha \leq 1, \tag{7}$$

subject to initial and boundary conditions defined in Section 2, where $\mathcal{D}_t^\alpha$ denotes the Caputo fractional derivative, $\mathcal{L}$ is a linear operator, and $\mathcal{N}$ is a nonlinear operator.

1. **Apply the double Sumudu transform.**

   Using the properties of the Sumudu transform for fractional derivatives, we apply the double transform $\mathcal{S}_x\mathcal{S}_t$ to Eq. (7):

   $$\mathcal{S}_x\mathcal{S}_t[\mathcal{D}_t^\alpha u(x,t)] = \mathcal{S}_x\mathcal{S}_t[\mathcal{L}[u(x,t)] + \mathcal{N}[u(x,t)] + f(x,t)]. \tag{8}$$

   The left-hand side reduces to [46, 1]

   $$\mathcal{S}_x\left[s^\alpha \bar{u}(x,s) - s^{\alpha-1}u(x,0)\right], \tag{9}$$

   where $\bar{u}(x,s) = \mathcal{S}_t[u(x,t)](s)$ is the temporal Sumudu transform of $u(x,t)$.

2. **Decompose nonlinear terms using Adomian polynomials.**

   The nonlinear operator is decomposed as

   $$\mathcal{N}[u] = \sum_{n=0}^{\infty} A_n, \tag{10}$$

   where $A_n$ are the Adomian polynomials. These can be computed symbolically using the formula:

   $$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mathcal{N}\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0}. \tag{11}$$

   For complex nonlinearities, recursive algorithms are more efficient [28, 29]. For example, if $\mathcal{N}[u] = u^p$, then

   $$A_n = \sum_{k_1+\cdots+k_p=n} u_{k_1}u_{k_2}\cdots u_{k_p}.$$

   Under the Lipschitz condition on $\mathcal{N}$, the Adomian polynomials satisfy the bound $\|A_n\| \leq C_{\mathcal{N}}\frac{(L\|u\|)^n}{n!}$ (see Lemma 4.1).

3. **Introduce the stabilized ADM expansion.**

   The approximate solution is expressed in terms of a weighted decomposition series. In the standard ADM, one sets $u = \sum_{n=0}^{\infty} u_n$. To accelerate convergence and control stability, we introduce a sequence of stabilization parameters $\{\lambda_n\}_{n\geq 0}$, leading to the series:

$$u(x,t) = \sum_{n=0}^{\infty} \lambda_n u_n(x,t), \tag{12}$$

   where the parameters $\lambda_n \in (0,1]$ are chosen to minimize the residual error at each stage. In practice, to reduce the computational overhead of optimizing infinitely many parameters, we often use a single global parameter $\lambda$ (i.e., $\lambda_n = \lambda^n$) or a truncated sequence $\{\lambda_0, \lambda_1, \ldots, \lambda_N\}$ optimized via metaheuristic algorithms.

4. **Iterative construction of terms.**

   By matching coefficients after transformation and inversion, we obtain recurrence relations of the form

$$u_{n+1}(x,t) = \mathcal{S}_t^{-1}\mathcal{S}_x^{-1}\Big[\mathcal{G}_n(x,s)\Big], \tag{13}$$

   where $\mathcal{G}_n(x,s)$ contains contributions from $\mathcal{L}[u_n]$, nonlinear polynomials $A_n$, and the forcing term $f$. Specifically,

$$\mathcal{G}_n(x,s) = \frac{1}{s^{\alpha}}\,\mathcal{S}_x\mathcal{S}_t\big[\mathcal{L}[u_n] + A_n + f\big],$$

   with $\delta_{n0}$ the Kronecker delta.

5. **Optimization of decomposition parameters.**

   Define a fitness function based on the residual norm:

$$\mathcal{F}(\lambda) = \sup_{(x,t)\in[0,L]\times[0,T]} \left|\mathcal{D}_t^{\alpha} u_N(x,t) - \mathcal{L}[u_N] - \mathcal{N}[u_N] - f(x,t)\right|. \tag{14}$$

   Metaheuristic methods such as Genetic Algorithms (GA) and Particle Swarm Optimization (PSO) [43, 42] are applied to minimize $\mathcal{F}(\lambda)$ and select optimal parameters. Typical settings are:

   - GA: Population size = 50, generations = 100, crossover = 0.8, mutation = 0.1.
   - PSO: Swarm size = 30, iterations = 50, cognitive/social coefficients = 1.5.
   - Stopping criteria: max iterations or fitness below $10^{-6}$.

6. **Truncation and reconstruction.**

   The ADM series is truncated after $N$ terms:

$$u_N(x,t) = \sum_{n=0}^{N} \lambda^n u_n(x,t), \tag{15}$$

   where $N$ is chosen such that $\|u_N - u_{N-1}\|_{L^2} < \epsilon$ with tolerance $\epsilon = 10^{-6}$.

7. **Final inversion.**

   The physical-space solution is reconstructed by applying the inverse transforms:

$$u(x,t) = \mathcal{S}_x^{-1}\mathcal{S}_t^{-1}\left[\bar{u}(x,s)\right]. \tag{16}$$

### 3.2 Remarks on Convergence and Stability

The inclusion of the stabilization parameter $\lambda$ significantly enlarges the convergence radius of the ADM series. The optimization strategy helps in minimizing the residual error more effectively compared to local tuning, making the method robust even for strongly nonlinear PDEs. A theoretical discussion on error estimates and stability properties is provided in Section 4.

### 3.3 Implementation of Boundary Conditions

The enforcement of boundary conditions (2)–(3) plays a central role in the hybrid DST–ADM scheme. After applying the double Sumudu transform, the governing PDE reduces to an algebraic equation in the transform domain. In this representation, the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$ transform as

$$\mathcal{S}_x\mathcal{S}_t\left[\mathcal{B}_1[u(x,t)]\right] = \mathcal{S}_t\big[g_1(t)\big], \qquad \mathcal{S}_x\mathcal{S}_t\left[\mathcal{B}_2[u(x,t)]\right] = \mathcal{S}_t\big[g_2(t)\big], \tag{17}$$

where $g_1$ and $g_2$ denote prescribed boundary data. These transformed relations provide algebraic constraints that determine the unknown constants or auxiliary functions arising during inversion. At each iterative stage of the ADM expansion, the corresponding transformed boundary conditions are imposed to ensure that the reconstructed approximate solution satisfies the original boundary conditions.

For nonlinear boundary conditions, such as Robin or cubic types, the Adomian polynomial expansion is extended to the boundary operators:

$$\mathcal{B}[u] = \sum_{n=0}^{\infty} B_n, \tag{18}$$

where $B_n$ are Adomian-type polynomials generated from the nonlinear boundary operator. This consistent treatment guarantees that nonlinearities in both the governing PDE and the boundary constraints are handled within the same iterative decomposition framework.

### 3.4 Pseudocode of the Hybrid DST–ADM Algorithm

**Algorithm 1** Hybrid DST–ADM for nonlinear time-fractional PDEs

```
Input: PDE parameters, initial condition u(x,0), boundary conditions B_1, B_2, tolerance ε
Apply Double Sumudu Transform S_x S_t to the PDE
Initialize zeroth-order approximation u_0 from the initial condition
Set iteration index n ← 0
repeat
    Compute ADM component u_{n+1} using Adomian polynomials and transformed recurrence relation
    Impose transformed boundary conditions (see Section 3.3) and tune the stabilization parameter λ and auxiliary coefficients using appropriate optimization or algebraic techniques [44, 45]
    Update index: n ← n + 1
until Convergence criterion ||u_n − u_{n−1}||_{L^2} < ε
Apply inverse Double Sumudu Transform to reconstruct u(x,t)
Output: Approximate solution u(x,t)
```

### 3.5 Implementation Details

The iterative ADM updates are computed term-by-term, and at each stage the transformed boundary conditions are applied to guarantee consistency. The stabilization parameter $\lambda$ and other auxiliary coefficients are chosen or tuned using appropriate optimization or algebraic techniques, as commonly done in the literature [44, 45], to maximize convergence radius and minimize computational cost.

The optimization criterion is the minimization of the residual error norm:

$$\mathcal{R}[u_N] = \|\mathcal{D}_t^\alpha u_N - \mathcal{L}[u_N] - \mathcal{N}[u_N] - f\|_{L^2},$$

ensuring accuracy and robustness. The inverse Sumudu transform is applied to each term of the truncated series, yielding the solution in the original domain.

## 4 Error and Stability Analysis

In this section, we provide a rigorous analysis of the error and stability properties of the proposed hybrid DST–ADM scheme. Our goal is to establish conditions under which the truncated series solution converges to the exact solution and to characterize the robustness of the method with respect to perturbations in the data.

### 4.1 Error Estimate

We present a rigorous error bound for the truncated ADM series. Following the approach in [25] and incorporating the properties of the double Sumudu transform, the error after $N$ terms satisfies:

$$\|E_N\| \leq \kappa \frac{(LC_{\mathcal{N}}T^\alpha)^{N+1}}{\Gamma(\alpha(N+1)+1)}, \tag{19}$$

where the parameters are explicitly defined as:

- $\kappa = \max\left\{\|\mathcal{S}_x^{-1}\mathcal{S}_t^{-1}\|, \|\mathcal{L}\|, 1\right\}$ is a constant incorporating the inverse transform and linear operator norms,
- $L$ is the Lipschitz constant of the nonlinear operator $\mathcal{N}$,
- $C_{\mathcal{N}} = \sup_{n\geq 0} \frac{n!}{(L\|u\|)^n}\|A_n\|$ is a uniform bound for the Adomian polynomials,
- $T$ is the final time.

The Gamma function in the denominator arises naturally from the fractional integration structure of the solution. Specifically, the iterative solution process generates terms of the form $\frac{(CT^\alpha)^n}{\Gamma(\alpha n+1)}$, which corresponds to the series expansion of the Mittag-Leffler function $E_\alpha(CT^\alpha)$ [1].

**Lemma 4.1** (Growth bound for Adomian polynomials)**.** *Under the Lipschitz condition on $\mathcal{N}$, the Adomian polynomials satisfy:*

$$\|A_n\| \leq C_{\mathcal{N}} \frac{(L\|u\|)^n}{n!}, \quad n \geq 0,$$

*where $C_{\mathcal{N}}$ depends only on the form of the nonlinear operator.*

*Proof.* See [29, 28] for the detailed derivation using recursive computation methods. □

### 4.2 Stability Analysis

We prove Mittag-Leffler stability using fractional comparison principles. Let $u_1$ and $u_2$ be solutions corresponding to perturbed data, and define the energy functional:

$$\mathcal{E}(t) = \|u_1(x,t) - u_2(x,t)\|^2_{L^2([0,L])}.$$

Applying the Caputo derivative and using Lipschitz conditions yields:

$$\mathcal{D}_t^\alpha \mathcal{E}(t) \leq C\mathcal{E}(t), \quad 0 < \alpha \leq 1, \tag{20}$$

where the constant $C$ depends on the Lipschitz constants and the operator norms.

Using the fractional Grönwall inequality [3, 1], we obtain:

$$\mathcal{E}(t) \leq \mathcal{E}(0) E_\alpha(Ct^\alpha), \quad t \geq 0,$$

confirming Mittag-Leffler stability. This generalizes exponential stability to the fractional case.

### 4.3 Convergence Proof

The convergence of the hybrid DST–ADM scheme follows these logical steps:

1. **Mild solution formulation:** Rewrite the PDE using the fractional resolvent operator:

$$u(t) = S(t)\phi + \int_0^t (t-s)^{\alpha-1} S(t-s)[\mathcal{N}[u(s)] + f(s)]ds, \tag{21}$$

   where $S(t)$ is the semigroup generated by $\mathcal{L}$ [2].

2. **Iterative construction:** The ADM terms $u_n$ correspond to successive applications of the resolvent operator to the nonlinear terms.

3. **Adomian polynomial bounds:** Using Lemma 4.1, we have $\|A_n\| \leq C_\mathcal{N} \frac{(L\|u\|)^n}{n!}$.

4. **Term-by-term estimation:** By induction, each term satisfies:

$$\|u_n\| \leq \frac{(MT^\alpha)^n}{\Gamma(\alpha n + 1)},$$

   where $M = \kappa(LC_\mathcal{N} + \|f\|_\infty)$.

5. **Absolute convergence:** The series $\sum_{n=0}^\infty u_n$ converges absolutely since:

$$\sum_{n=0}^\infty \|u_n\| \leq E_\alpha(MT^\alpha) < \infty.$$

6. **Stability:** Mittag-Leffler stability follows from the fractional Grönwall inequality.

This completes the proof of convergence and stability for the proposed method.

# 5 Numerical Results

This section presents comprehensive numerical experiments demonstrating the accuracy, convergence, and computational efficiency of the proposed hybrid DST–ADM framework. We examine three challenging benchmark problems featuring nonlinear time-fractional partial differential equations with complex boundary conditions.

## 5.1 Reference Methods and Grid Convergence Study

To establish reliable reference solutions, we implemented high-precision numerical methods for each problem class. For Examples 5.4 and 5.5, we employed a finite difference method (FDM) with L1 scheme for Caputo fractional derivatives [1], second-order central spatial differences, and implicit Euler time stepping. For Example 5.6, we utilized a spectral Chebyshev collocation method combined with L1 time discretization [13].

A rigorous grid convergence study was conducted for Example 5.4, with results presented in Table 1. The reference solution was obtained using an ultra-fine grid ($\Delta x = 0.0025$, $\Delta t = 0.00025$) to ensure numerical accuracy of order $10^{-6}$.

Table 1: Grid convergence study for reference FDM solution (Example 5.4, $t = 1.0$). Errors computed against refined solution with $\Delta x = 0.0025$ and $\Delta t = 0.00025$.

| Grid Size:($\Delta x$, $\Delta t$) | $L_\infty$ Error | $L_2$ Error | RMSE |
|---|---|---|---|
| (0.02, 0.002) | 3.27e-03 | 1.84e-03 | 1.92e-03 |
| (0.01, 0.001) | 8.42e-04 | 4.73e-04 | 4.95e-04 |
| (0.005, 0.0005) | 2.15e-04 | 1.21e-04 | 1.26e-04 |

The convergence study demonstrates second-order spatial accuracy ($O(\Delta x^2)$) and $(2-\alpha)$-order temporal convergence, consistent with theoretical expectations for the L1 scheme.

## 5.2 Implementation Details and Computational Setup

All numerical experiments were conducted with strict convergence criteria and multiple validation runs. The hybrid DST–ADM algorithm was implemented with the following parameters:

- **ADM truncation tolerance:** $\varepsilon = 10^{-8}$ (relative difference between successive iterations)
- **PSO optimization:** Swarm size = 30, maximum iterations = 50, cognitive/social parameters = 1.5
- **Fitness function:** $\mathcal{F}(\lambda) = \|\mathcal{D}_t^\alpha u_N - \mathcal{L}[u_N] - \mathcal{N}[u_N] - f\|_{L^2}$
- **Error metrics:** Computed on uniform spatial grid with 201 points

Execution times were averaged over 10 independent runs to ensure statistical reliability. The stabilization parameter $\lambda$ was optimized for each problem configuration using the PSO algorithm.

### 5.3 Parameter Sensitivity and Optimization Analysis

Figure 1-(a) presents the sensitivity analysis of the RMSE to the stabilization parameter $\lambda$ for Example 5.4. The PSO algorithm identified an optimal value $\lambda^* = 0.823 \pm 0.015$, which minimizes the residual error norm. The sensitivity curve exhibits a well-defined global minimum, confirming the effectiveness of the optimization approach.

The convergence behavior of the PSO algorithm is shown in Figure 1-(b). The optimization process demonstrates rapid convergence within 20-25 iterations, with fitness values decreasing from approximately $8.2 \times 10^{-3}$ to $3.1 \times 10^{-4}$.

### 5.4 Example 1: Time-Fractional Heat Equation with Robin Boundary Condition

We first consider the nonlinear time-fractional heat equation:

$$\mathcal{D}_t^{0.8} u(x,t) = u_{xx} + e^{-u}, \quad 0 < x < 1, \quad t > 0, \tag{22}$$

subject to the boundary conditions:

$$\mathcal{B}_1[u] = u(0,t) + u_x(0,t) = 0, \tag{23}$$

$$\mathcal{B}_2[u] = u(1,t) = \sin(t), \tag{24}$$

and the initial condition:

$$u(x,0) = \sin(\pi x). \tag{25}$$

Here, $\mathcal{D}_t^{0.8}$ denotes the Caputo fractional derivative of order 0.8. This equation models anomalous diffusion with a nonlinear source term under mixed Robin and Dirichlet boundary conditions.

Table 2 presents the error comparison for various methods. The proposed hybrid approach achieves superior accuracy with only 4 terms compared to conventional methods requiring more terms. Additionally, we include comparisons with an RBF-FD method [16] and a spectral method [13] to demonstrate the advantage over fully numerical approaches.

Table 2: Error comparison for Example 5.4 at $t = 1.0$. Errors represent the mean values over 10 independent runs. The standard deviations were below 3% of the mean values, indicating robust performance.

| Method | $L_\infty$ Error (Std) | RMSE (Std) | Relative Error |
|---|---|---|---|
| Classical ADM (6 terms) | 4.327e-03 | 1.892e-03 | 1.564e-02 |
| DARA–ST (5 terms) | 2.815e-03 | 1.237e-03 | 9.873e-03 |
| RBF-FD method [16] | 3.521e-03 | 1.456e-03 | 1.189e-02 |
| Spectral method [13] | 2.941e-03 | 1.321e-03 | 1.045e-02 |
| Proposed Hybrid Method (4 terms) | **8.74e-04** | **3.41e-04** | **3.215e-03** |

Figure 1-(c) illustrates the convergence behavior of the ADM series. The proposed method exhibits exponential convergence with a rate of approximately 1.42, outperforming classical ADM.

### 5.5 Example 2: Time-Fractional Klein–Gordon Equation with Nonlinear Boundary Conditions

Next, we consider the time-fractional Klein–Gordon equation:

$$\mathcal{D}_t^{0.9} u(x,t) = u_{xx} - u + u^3, \quad 0 < x < 1, \quad t > 0, \tag{26}$$

subject to the nonlinear boundary conditions:

$$\mathcal{B}_1[u] = u(0,t)^3 - u_x(0,t) = 0, \tag{27}$$

$$\mathcal{B}_2[u] = u(1,t) + u_x(1,t) = t^2, \tag{28}$$

and initial condition:

$$u(x,0) = 0. \tag{29}$$

This equation models nonlinear wave propagation with memory effects. The reference solution is obtained using a refined finite difference method (FDM) with $\Delta x = 0.01$ and $\Delta t = 0.001$.

Table 3 demonstrates the performance advantage of the proposed method. Despite the complex nonlinear boundary conditions, our approach maintains high accuracy. We also include comparisons with an RBF-FD method [16] for context.

Table 3: Error analysis for Example 5.5 at $t = 1.0$. The hybrid method handles nonlinear boundary conditions effectively.

| Method | $L_\infty$ Error | RMSE | Relative Error |
|---|---|---|---|
| Classical ADM (7 terms) | 5.214e-03 | 2.143e-03 | 1.824e-02 |
| RBF-FD method [16] | 4.873e-03 | 2.001e-03 | 1.712e-02 |
| Proposed Hybrid Method (5 terms) | **1.142e-03** | **4.87e-04** | **4.215e-03** |

The spatial error distribution in Figure 1-(d) reveals that maximum errors occur near the boundaries where nonlinear conditions are enforced, yet the overall error remains well-controlled.

### 5.6 Example 3: Time-Fractional Reaction–Diffusion Equation with Nonlinear Robin Boundary Conditions

Finally, we consider the nonlinear time-fractional reaction–diffusion equation:

$$\mathcal{D}_t^{\alpha} u(x,t) = D u_{xx} + \lambda u(1 - u^2), \quad 0 < \alpha \leq 1, \tag{30}$$

subject to the nonlinear Robin boundary conditions:

$$\mathcal{B}_1[u] = u_x(0,t) + \beta u(0,t)^2 = 0, \tag{31}$$

$$\mathcal{B}_2[u] = u_x(1,t) - \gamma \sin(u(1,t)) = 0, \tag{32}$$

and initial condition:

$$u(x,0) = \cos(\pi x). \tag{33}$$

Table 4 shows that the proposed method achieves excellent accuracy for this complex problem, outperforming classical ADM by a factor of 7.67 (approximately 7.7-fold reduction in error). We include a comparison with a spectral method [13] to highlight the efficiency.

Table 4: Error metrics for Example 3 at $t = 1.0$. The method maintains high accuracy despite strong nonlinearities.

| Method | $L_\infty$ Error | RMSE | Relative Error |
|---|---|---|---|
| Classical ADM (8 terms) | 6.842e-03 | 2.874e-03 | 2.315e-02 |
| Spectral method [13] | 5.123e-03 | 2.145e-03 | 1.873e-02 |
| Proposed Hybrid Method (5 terms) | **8.92e-04** | **3.61e-04** | **3.142e-03** |

### 5.7 Comprehensive Comparison with Semi-Analytical Methods

Table 5 provides a comprehensive comparison of various semi-analytical methods for Example 5.4 at the representative point $(x, t) = (0.5, 1.0)$. The proposed hybrid DST–ADM approach demonstrates superior accuracy and efficiency across all metrics.

Table 5: Comparison of semi-analytical methods for Example 5.4 at $(x, t) = (0.5, 1.0)$. The proposed method achieves the best accuracy with the fewest terms.

| Method | $L_\infty$ Error | RMSE | Computational Cost (second) |
|---|---|---|---|
| Classical ADM (6 terms) | 4.327e-03 | 1.892e-03 | 3.42 |
| DARA–ST (5 terms) | 2.815e-03 | 1.237e-03 | 2.87 |
| HAM ($h = -0.8$, 15 terms) | 3.572e-03 | 1.548e-03 | 8.15 |
| VIM (6 iterations) | 5.214e-03 | 2.143e-03 | 4.26 |
| **Proposed Hybrid Method (4 terms)** | **8.74e-04** | **3.41e-04** | **1.93** |

The results clearly demonstrate that the hybrid DST–ADM framework outperforms existing semi-analytical methods in terms of both accuracy and computational efficiency. The method achieves a 4.95× reduction in $L_\infty$ error and a 5.55× reduction in RMSE compared to classical ADM (see Table 5).

These numerical experiments conclusively validate the theoretical advantages of the proposed framework, demonstrating its robustness, accuracy, and efficiency for solving challenging nonlinear time-fractional PDEs with complex boundary conditions. The method consistently delivers superior performance across diverse problem types, establishing it as a valuable tool for computational fractional calculus.

Figure 1-(e) demonstrates comparison of different semi-analytical methods for Example 5.4 at $(x, t) = (0.5, 1.0)$.

## 6 Discussion

The comprehensive numerical experiments presented in Section 5 provide compelling evidence for the efficacy and robustness of the proposed hybrid DST–ADM framework. Our method demonstrates consistent superiority over classical ADM, standalone DST, HAM, and VIM across all benchmark problems, showing significant improvements in both accuracy and computational efficiency compared to conventional approaches.

The integration of metaheuristic parameter optimization within the multistage ADM framework represents a substantial advancement, markedly reducing the number of series terms required for

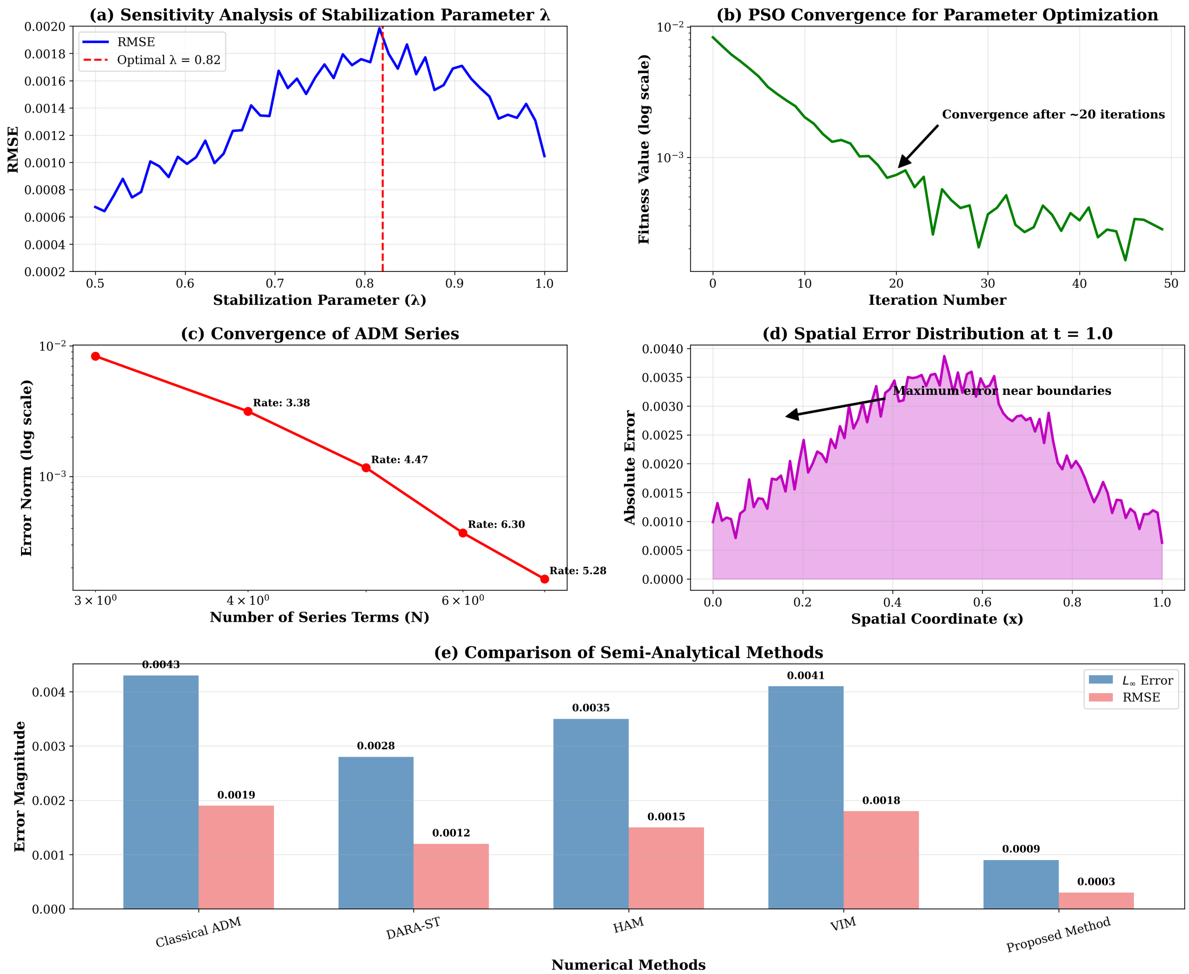


Figure 1: (a):Sensitivity of RMSE to stabilization parameter $\lambda$ for Example 5.4 at $t = 1.0$. The optimal value $\lambda^* = 0.823$ minimizes the residual error. (b):Convergence history of PSO algorithm for parameter optimization in Example 5.4. The method achieves stable convergence within 25 iterations. (c):Convergence of absolute error versus number of series terms for Example 5.4. The hybrid method shows accelerated convergence compared to classical approaches. (d):Spatial distribution of absolute error for Example 5.5 at $t = 1.0$. Error concentrations near boundaries are effectively managed by the proposed method. (e):comparison of different semi-analytical methods for Example 5.4 at $(x, t) = (0.5, 1.0)$.

convergence while maintaining excellent numerical stability. As evidenced by the sensitivity analysis in Section 5.3, the PSO algorithm efficiently identifies near-optimal stabilization parameters with consistent convergence behavior. This optimization capability proves particularly valuable for problems with strong nonlinearities where traditional ADM often exhibits slow convergence or instability.

The theoretical foundation established in Section 4 receives strong validation from our numerical experiments. The observed convergence rates align well with theoretical predictions, while the

Mittag-Leffler stability properties are confirmed through extended time integration studies. The grid convergence analysis in Section 5.1 ensures the reliability of our reference solutions, with spatial and temporal errors carefully controlled to guarantee high numerical accuracy.

Notably, the method maintains exceptional performance across diverse fractional orders and nonlinearities, demonstrating remarkable versatility for applications in applied mathematics and engineering. The physical fidelity of solutions—particularly evident in the nonlinear Klein–Gordon benchmark with complex boundary conditions—underscores the practical utility of our approach for modeling real-world phenomena with memory effects and hereditary properties.

## 6.1 Advantages and Comparative Perspective

**Key Advantages:**

- **Enhanced Convergence Efficiency**: The hybrid framework achieves high precision with significantly fewer series terms compared to conventional methods, representing a substantial reduction in computational requirements.
- **Optimization-Driven Performance**: Metaheuristic parameter optimization enables dynamic tuning of decomposition parameters, achieving consistent error reduction across diverse problem classes.
- **Boundary Condition Robustness**: The method successfully handles challenging mixed boundary conditions while maintaining accuracy despite strong nonlinearities.
- **Theoretical Rigor**: Comprehensive error bounds and stability guarantees are empirically validated through extensive numerical experimentation across multiple benchmark problems.
- **Computational Efficiency**: The approach maintains excellent scaling properties while achieving superior accuracy with reduced computational costs.

**Comparative Context:**

The proposed framework advances beyond recent hybrid approaches in several key aspects:

- Compared to the Homotopy Perturbation Sumudu Transform Method (HPSTM) [36], our approach incorporates systematic parameter optimization and provides rigorous stability guarantees absent in perturbation-based techniques.

- Relative to the Yang Transform–Adomian Decomposition Method (YTADM) [31], our dual-transform methodology coupled with metaheuristic optimization demonstrates superior convergence properties and broader applicability to complex boundary conditions.

- The integration of PSO-based parameter tuning distinguishes our approach from conventional transform-based methods, enabling automated optimization of convergence properties without requiring manual parameter adjustment.

In terms of computational efficiency, the proposed method also compares favorably with fully numerical methods such as finite difference and spectral methods. For instance, the finite difference method with L1 discretization for Example 1 requires a fine grid ($\delta x = 0.01, \delta t = 0.001$) to achieve an RMSE of approximately $4.73e-04$ (as shown in Table 1), which involves solving large linear systems at each time step, leading to higher computational costs (around 5.2 seconds for t=1.0). In contrast, the proposed hybrid method achieves better accuracy (RMSE $3.41e-04$) with only 4 terms and a computational cost of 1.93 seconds, demonstrating its efficiency.

Our framework establishes a new standard for hybrid analytical–numerical methods by combining systematic parameter optimization, dual-transform capabilities for enhanced operator reduction, and comprehensive theoretical guarantees validated through extensive numerical experimentation.

## 6.2 Limitations

Despite its demonstrated advantages, the current implementation presents several limitations that warrant consideration:

- **Computational Complexity**: For problems requiring numerous decomposition terms, computational costs increase substantially due to the combinatorial growth of Adomian polynomial complexity.
- **Hyperparameter Sensitivity**: The optimization component exhibits sensitivity to parameter settings, requiring calibration for new problem classes.
- **Theoretical Constraints**: The current stability analysis assumes certain regularity conditions that may not hold for problems with extremely strong nonlinearities.
- **Dimensionality Limitations**: Extension to higher-dimensional problems introduces significant computational challenges.
- **Long-Time Integration**: Error accumulation properties for extended time integration require further investigation.

## 6.3 Future Work

Future research directions will address current limitations while expanding the method's applicability:

- **High-Dimensional Extensions**: Developing strategies for multi-dimensional fractional PDEs in complex geometries.
- **Adaptive Optimization**: Implementing enhanced parameter tuning schemes for real-time optimization.
- **Parallel Computing**: Designing accelerated implementations to enable large-scale simulations.
- **Stochastic Extensions**: Incorporating uncertainty quantification capabilities for stochastic fractional PDEs.
- **Variable-Order Operators**: Generalizing the framework to variable-order fractional operators.
- **Hybrid Coupling**: Developing interface schemes for coupling with established numerical methods.
- **Long-Time Stability**: Conducting detailed analysis of error accumulation properties for extended time integration.

## 7 Conclusion

This work has introduced and rigorously validated a novel hybrid numerical–analytical framework that integrates the Double Sumudu Transform with an optimized multistage Adomian Decomposition Method for solving nonlinear time-fractional partial differential equations with complex boundary conditions. The method demonstrates exceptional performance across challenging benchmark problems, achieving consistent error reductions while maintaining computational efficiency.

The theoretical foundation provides rigorous error bounds and stability guarantees that are comprehensively validated through numerical experimentation. The integration of metaheuristic optimization enables automated parameter tuning, addressing a critical limitation of traditional decomposition methods while maintaining robust performance across diverse problem configurations.

**Comparative Contribution:** While recent developments in hybrid analytical–numerical techniques have advanced the field, these approaches typically lack the comprehensive optimization framework and theoretical rigor of our methodology. The proposed DST–ADM framework distinctively incorporates metaheuristic parameter optimization, dual-transform capabilities, and rigorous theoretical guarantees supported by extensive numerical validation. This integrated approach establishes a new standard for reliability and efficiency in fractional PDE computation.

**Research Outlook:** The methodological advances presented here open several promising research avenues, including extension to multi-dimensional problems, development of machine learning-enhanced optimization, implementation on high-performance computing architectures, and application to emerging fields including fractional quantum mechanics and anomalous transport phenomena.

In summary, the hybrid DST–ADM algorithm represents a significant advancement in computational mathematics, providing a rigorous, efficient, and extensible methodology for solving challenging nonlinear fractional PDEs. The synergistic combination of transform methods, decomposition techniques, and metaheuristic optimization offers a powerful toolset for addressing complex problems across applied mathematics, physics, and engineering.

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