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Research Article

A Novel Bernoulli Operational Matrix Method for Numerical Solution of Nonlinear Multi-term Variable-order Fractional Differential Equations

Hamid Reza Khodabandehlo*
Department of Applied Mathematics, Imam Khomeini International University

Abstract

Abstract: This article presents a numerical method for addressing a category of nonlinear multi-term variable-order fractional derivative equations. The approach relies on creating a Bernoulli operational matrix (BOM) for fractional variable-order derivatives. This technique is utilized to address a category of these equations, transforming the original problem into a system of algebraic equations amenable to numerical solutions. Comprehensive and thorough numerical tests are provided to demonstrate the precision, applicability, and effectiveness of the proposed method, along with the flexibility of this strategy. The numerical outcomes from this approach are contrasted with the exact solution. A comparison of the results from this scheme with the exact solution shows that the new approach is efficient, yielding high-accuracy approximate solutions even with a limited number of basis functions, and in cases where the problem's solution lacks infinite differentiability, offering superior results and fewer basis functions compared to advanced methods. Additionally, numerous physical application issues involving multi-term variable-order fractional differential equations, such as the damped mechanical oscillator problem and the Bagley-Torvik equation, can be addressed using the proposed method.

Keywords: Fractional variable-order differential equations, Nonlinear multi-term differential equations, Bernoulli operational matrix, Caputo differential operator.
2020 Mathematics Subject Classification: 65M99

1 Introduction

Over the past few decades, fractional calculus has garnered interest from numerous researchers across various disciplines including blood flow dynamics, biophysics, chemistry, physics, ongoing thermodynamic variations, the electro-dynamics of complex materials, capacitor principles, polymer rheology, dynamic systems, experimental data fitting, and more ([1, 2, 3, 4, 5] and references therein). The growing advancement of suitable and effective techniques for addressing fractional

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*Corresponding author: khodabandelo.hamidreza@yahoo.com

differential equations (FDEs) has generated heightened interest among scholars in this area. Owing to the various uses of these equations, several techniques have been developed to solve them, including Fractional Adams-Moulton methods [6], fractional linear multi-steps methods [7], trapezoidal methods [8], and many others.

A recent extension of the fractional calculus theory permits the order of derivatives to vary with time, meaning it can be nonconstant or of variable order. The fractional variable-order calculus is viewed as a standard filter that provides a robust mathematical framework for describing dynamic issues in a complex manner [9, 10]. Besides this crucial introduction of fractional applications, it is undeniable that scientific research has revolutionized many physical and mathematical issues represented by nonlinear models (refer to, for example, [11, 12, 13, 14, 15, 16]). Specifically, various uses of variable-order fractional calculus are observed in engineering mechanics; an instance of variable-order fractional operators modeling a material's microscopic structure [17], an application of the Riesz-Caputo fractional derivative of space-dependent order in continuum elasticity[18], [19] for the nonlinear viscoelastic characteristics of fractional systems with changing time-dependent fractional order.

Finding exact solutions for the majority of FDEs is not straightforward, leading to the need for analytical and numerical techniques to be employed, On the other hand, it is known that acquiring analytical solutions to these equations is quite challenging. Therefore, in many cases, the precise solution remains unknown, and it is necessary to pursue a numerical estimation. Consequently, numerous investigators have devised and advanced numerical techniques to facilitate to acquire estimated solutions for this category of equations. For example, in [20], Numerical solution of nonlinear delay differential equations of fractional variable-order using a novel shifted Jacobi operational matrix is presented. In [21, 22, 23], A novel shifted Jacobi operational matrix method for linear and nonlinear multi-terms delay differential equations of fractional variable-order with periodic and anti-periodic conditions and for Nonlinear Fractional Variable-Order Differential Equation with Proportional Delays is proposed. Reference [24] used A novel Jacobi operational matrix for numerical solution of multi-term variable-order fractional differential equations. In [25], numerical solution of variable-order FDEs Using Bernoulli polynomials is peresented.

Recently, Bernoulli polynomials have demonstrated their strength as a mathematical tool for addressing involving a range of dynamic issues, such as variable-order FDEs [25], numerically addressing high-order Fredholm integro-differential equations [26], pantograph equations [27], PDEs [28], linear Volterra and nonlinear Volterra-Fredholm-Hammerstein IEs [29], alongside optimal control challenges [30].

At present, the primary objective of this article is to extend the classical polynomials in the foundation of the solution. We propose a Novel Operational Matrix technique that relies on Bernoulli polynomials to compute the solution of nonlinear multi-term variable FDEs numerically. This approach employs the $(NBOM)$ method to convert the primary problem into a set of algebraic equations that are solvable by well-known numerical methods.

Thus, we present a $(NBOM)$ for the derivatives of fractional variable-order for solving a category of $VFDE$s which are as follows:

$$D^{\mu(t)}z(t) = F\left(t,\, z(t),\, D^{\zeta_1(t)}z(t),\, D^{\zeta_2(t)}z(t),\, ...,\, D^{\zeta_n(t)}z(t)\right),\ 0 \le t \le T\,, \tag{1}$$

$$z(0) = z_0, \tag{2}$$

where $D^{\mu(t)}z(t)$ and $D^{\zeta_s}z(t)(s = 1, 2, 3, 4, \cdots, n)$ represent the Caputo's derivative of fractional-order variable.

Remark 1.1. If $\zeta_s(t)(s = 1, 2, 3, 4, \cdots, n)$ represent constants, then equations (1)-(2) will appear as follow:

$$\begin{aligned}&D^{\mu}z(t) = F\left(t,\ z(t),\ D^{\zeta_1}z(t),\ D^{\zeta_2}z(t),\ ...,\ D^{\zeta_n}z(t)\right),\ 0 \le t \le T\ ,\\ &z(0) = z_0.\end{aligned}$$

2 Preliminaries and Methods

In this part, we evalution several key and essential aspects of fractional calculus theory. Next, we highlight some significant characteristics of Bernoulli polynomials that assist us in formulating the proposed technique.

2.1 The fractional order derivative

Various definitions exist and are utilized for the fractional derivative, yet the three most common definitions of them are presented by Caputo, Grünwald-Letincov, and Riemann-Liouville. Due to the Caputo fractional derivative is the only model that behaves like the integer-order DE, in this article we use it.

Definition 2.1 (Caputo fractional derivatives of order ζ)**.** The ζ-order ($m-1 < \zeta \le m$) fractional derivatives of Caputo (right and left-sided) are presented as[31]:

$$\begin{aligned}D^{\zeta}_{-}z(t) &= \frac{(-1)^m}{\Gamma(m-\zeta)}\int_t^T \frac{z'(s)}{(s-t)^{\zeta-m+1}}ds,\\ D^{\zeta}_{+}z(t) &= \frac{1}{\Gamma(m-\zeta)}\int_0^t \frac{z'(s)}{(t-s)^{\zeta-m+1}}ds,\end{aligned} \tag{3}$$

that

$$D^{\zeta}_{+}t^{w} = \begin{cases} 0, & for\; w \in M_0\; and\; w < \lceil\zeta\rceil,\\ \dfrac{\Gamma(w+1)}{\Gamma(w-\zeta+1)}t^{w-\zeta}, & for\; w \in M_0\; and\; w > \lceil\zeta\rceil,\end{cases} \tag{4}$$

and

$$D^{\zeta}_{-}(T-t)^{w} = \begin{cases} 0, & for\; w \in M_0\; and\; w < \lceil\zeta\rceil,\\ \dfrac{(-1)^w\Gamma(w+1)}{\Gamma(w-\zeta+1)}(T-t)^{w-\zeta}, & for\; w \in M_0\; and\; w > \lceil\zeta\rceil,\end{cases} \tag{5}$$

where $M_0 = \{0, 1, 2, \cdots\}$ and $\lceil . \rceil$ is the ceiling function. Also

$$D^{\zeta}_{\pm}(\gamma\varphi(t)\ +\ \delta\phi(t))\ =\ \gamma D^{\zeta}_{\pm}(\varphi(t))\ + \delta D^{\zeta}_{\pm}(\phi(t)),$$

where $\gamma$and δ are constants.

Definition 2.2 (Caputo variable fractional derivatives of order $\zeta(t)$)**.** The Caputo's derivative of fractional-order variable $\zeta(t)$ for $z(t) \in C^m[0,\, T]$ is expressed as [32, 33]:

$$D^{\zeta(t)}z(t) = \frac{1}{\Gamma(1-\zeta(t))}\int_{0^+}^{t}\frac{z'(g)}{(t-g)^{\zeta(t)}}dg + \frac{z(0^+)-z(0^-)}{\Gamma(1-\zeta(t))}t^{-\zeta(t)}. \tag{6}$$

In the initial moment and for $0 < \zeta(t) < 1$, possess:

$$D^{\zeta(t)}z(t) = \frac{1}{\Gamma(1-\zeta(t))}\int_{0^+}^{t}\frac{z'(g)}{(t-g)^{\zeta(t)}}dg, \tag{7}$$

and, for the constants $k$and r, we possess

$$D^{\zeta(t)}(k\, z_1(t) + br\, z_2(t)) = k\, D^{\zeta(t)}z_1(t) + r\, D^{\zeta(t)}z_2(t)\,. \tag{8}$$

Based on Eq.(6), for a fixed C we will have:

$$D^{\zeta(t)}C \;=\; 0. \tag{9}$$

On the other hand

$$D^{\zeta(t)}t^k = \begin{cases} 0, & for\, k \;=\; 0, \\ \dfrac{\Gamma(k+1)}{\Gamma(k+1-\zeta(t))}t^{k-\zeta(t)}, & for\, k = 1, 2, \cdots\,. \end{cases} \tag{10}$$

2.2 Bernoulli Polynomials and their properties

The polynomials of Bernoulli form a set of independent polynomials that constitute a complete basis for all square-integrable function spaces over the interval $[0, 1]$ (known as the space $L^2[0, 1]$).

Let $B_n(s)$ represents the $n-$th degree Bernoulli polynomial in s, is described as follows [25, 34]:

$$B_n(s) \;=\; \sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right) b_{n-i}\, s^j, \tag{11}$$

where $b_j, j = 0, 1, ..., n$, denote the numbers of Bernoulli that found in the series expansion of trigonometric functions [25, 35] and can be characterized using the subsequent identification:

$$\frac{s}{e^s - 1} = \sum_{j=0}^{\infty} b_j \frac{s^j}{j!}, \tag{12}$$

thus

$$\begin{aligned} &B_0(s) = 1, \\ &B_1(s) = s - \frac{1}{2}, \\ &B_2(s) = s^2 - s + \frac{1}{6}, \\ &B_3(s) = s^3 - \frac{3}{2}s^2 + \frac{1}{2}s, \\ &B_4(s) = s^4 - 2s^3 + s^2 - \frac{1}{30}, \end{aligned}$$

the initial five Bernoulli polynomials.

The subsequent property holds for Bernoulli polynomials [35]:

$$\int_0^1 B_k(s)\, B_m(s) = (-1)^{-1+k} \frac{m!\, k!}{(m+k)!}\, b_{m+k},\ \ m,k \geq 1.$$

Conversely, the Bernoulli polynomials can be conveniently produced using the subsequent recursive relation

$$\sum_{j=0}^{n-1} \left(\begin{array}{c} n \\ j \end{array} \right) B_j(s) = ns^{n-1}\ ,\ j = 2, 3, \cdots . \tag{13}$$

We remember that the benefits of Bernoulli polynomials in estimating any unknown arbitrary function, compared to certain traditional orthogonal polynomials, are:

I. The operational matrix of derivatives for Bernoulli contains fewer nonzero elements compared to that of some shifted classical orthogonal polynomials. The nonzero entries of the Bernoulli operational matrix exist solely in the first subdiagonal. In contrast, the shifted Jacobi and Chebyshev polynomials form a strictly lower triangular matrix, as noted in[36, 37].

II. The Bernoulli polynomials contain fewer terms than the several traditional orthogonal polynomials. For instance, the sixth Bernoulli polynomial contains five terms, whereas the sixth shifted Chebyshev polynomial consists of seven terms, and this disparity will grow as the degree increases. Thus, when approximating any arbitrary function, we utilize more CPU time by employing classical orthogonal polynomials in contrast to Bernoulli polynomials; for further reading refer to [38].

III. The coefficients of separate terms in Bernoulli polynomials are less than such coefficients in the traditional orthogonal polynomials. As the calculation errors in the product are tied to the coefficients of separate terms, employing Bernoulli polynomials reduces these errors.

Therefore, considering the above-mentioned issues and due to its high accuracy and easy implementation, using Bernoulli operational matrix method is economical.

3 Approximating function using Bernoulli polynomials

Assume the function $z(t)$ is a random square integrable function ($z(t) \in L^2[0,1]$), consequently, it can be stated in the following form[25]:

$$z(t)\ =\ \sum_{j=0}^{\infty} f_j B_j(t), \tag{14}$$

where f_j (the coefficients of the series) are obtained using the formula presented in [29, 30] as follows:

$$f_j\ =\ \frac{1}{j!} \int_0^1 \frac{d^j z(t)}{dt^j} dt. \tag{15}$$

Thus, we can approximate the solution utilizing $(M+1)$-terms of the given series in Eq. (14) and we shall possess

$$z(t) \simeq z_M(t) = \sum_{j=0}^{M} f_j B_j(t) = F^T \Phi_M(t), \tag{16}$$

where $F = [f_0, f_1, f_2, f_3 \cdots, f_M]^T$, and $\Phi_M(t) = [B_0(t), B_1(t), B_2(t), B_3(t), \cdots, B_M(t)]^T$.

In this context, we presume that

$$S(t) = \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \\ t^5 \\ \cdots \\ t^M \end{bmatrix}. \tag{17}$$

According to Eq.(16)

$$\Phi_M(t) = \begin{bmatrix} B_0(t) \\ B_1(t) \\ \vdots \\ B_i(t) \\ B_{i+1}(t) \\ \vdots \\ B_M(t) \end{bmatrix} = \begin{bmatrix} b_0 \\ \sum_{k=0}^{1} \begin{pmatrix} 1 \\ k \end{pmatrix} b_{1-k}\, t^k \\ \vdots \\ \sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} b_{i-k}\, t^k \\ \sum_{k=0}^{i+1} \begin{pmatrix} i+1 \\ k \end{pmatrix} b_{i-k+1}\, t^k \\ \vdots \\ \sum_{k=0}^{M} \begin{pmatrix} M \\ k \end{pmatrix} b_{M-k}\, t^k \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} \theta_{1,1} & \theta_{1,2} & \cdots & \theta_{1,M+1} \\ \theta_{2,1} & \theta_{2,2} & \cdots & \theta_{2,M+1} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{i,1} & \theta_{i,2} & \cdots & \theta_{i,M+1} \\ \theta_{i+1,1} & \theta_{i+1,2} & \cdots & \theta_{i+1,M+1} \\ \vdots & \vdots & \vdots & \vdots \\ \theta_{M+1,1} & \theta_{M+1,2} & \cdots & \theta_{M+1,M+1} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^i \\ t^{i+1} \\ \vdots \\ t^M \end{bmatrix} = \Theta S(t),$$

where Θ is a square matrix described as follows:

$$\theta_{l+1,k+1} = \begin{cases} \begin{pmatrix} l \\ k \end{pmatrix} b_{l-k},\ l \geq k, \\ \\ 0\,,\ otherwise, \end{cases} \tag{19}$$

for $0 \leq l, k \leq M$.

Then Θ as follows

$$\Theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{6} & -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ b_M & \binom{M}{1} b_{M-1} & \binom{M}{2} b_{M-2} & \binom{M}{3} b_{M-3} & \binom{M}{4} b_{M-4} & \cdots & 1 \end{bmatrix}_{(M+1)\times(M+1)}. \tag{20}$$

Hence, using Eq. (18), we get

$$S(t) = \Theta^{-1} \Phi_M(t). \tag{21}$$

4 Novel Bernoulli Polynomials Operational Matrix ($NBOM$)

Operational matrices, utilized across various fields of numerical analysis, address diverse issues of various kinds and subjects are particularly significant, including IEs, DEs, and integro-differential equations, partial and ordinaryFDEs [31, 39, 40, 41, 42, 43, 44, 45, 46, 47]. Now, we explore the ($NBOM$) of fractional order to assist in the computational solution of Eqs. (1), (2). Thus, we transform the original problem into a set of algebraic equations that can be solved using numerical techniques in collocation points.

Initially, we infer $D^{\mu(t)}\Phi_M(t)$ as follows:
based on the previous content, have: $\Phi_M(t) = \Theta S(t)$, so

$$D^{\mu(t)}\Phi_M(t) = D^{\mu(t)}(\Theta S(t)) = \Theta D^{\mu(t)} [1, t, \cdots, t^M]^T. \tag{22}$$

Combining Eqs.(10) and (22), in general, it gives:

$$\begin{aligned} D^{\mu(t)}\Phi_M(t) &= \Theta D^{\mu(t)}(S(t)) \\ &= \Theta\, [0, \frac{\Gamma(2)t^{(1-\mu(t))}}{\Gamma(2-\zeta_i(t))}, \frac{\Gamma(3)t^{(2-\mu(t))}}{\Gamma(3-\mu(t))}, \cdots, \frac{\Gamma(1+M)t^{(M-\mu(t))}}{\Gamma(1+M-\mu(t))}]^T \\ &= \Theta \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)t^{-\mu(t)}}{\Gamma(2-\mu(t))} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)t^{-\mu(t)}}{\Gamma(3-\mu(t))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(1+M)t^{-\mu(t)}}{\Gamma(1+M-\mu(t))} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^M \end{bmatrix} \qquad (23) \\ &= \Theta G(t) S(t), \end{aligned}$$

where

$$G(t)=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \dfrac{\Gamma(2)t^{-\mu(t)}}{\Gamma(2-\mu(t))} & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{\Gamma(3)t^{-\mu(t)}}{\Gamma(3-\mu(t))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\Gamma(1+M)t^{-\mu(t)}}{\Gamma(1+M-\mu(t))} \end{bmatrix}. \tag{24}$$

Using Eq. (21), then

$$D^{\mu(t)}\Phi_M(t) \ = \ \Theta G(t)\,\Theta^{-1}\Phi_M(t). \tag{25}$$

The operational matrix of $D^{\mu(t)}\Phi_M(t)$, is $\Theta G(t)\,\Theta^{-1}$.

Here, we estimate the variable-order fractional of the calculated function that obtained in Eq.(16) as follows

$$D^{\mu(t)}z(t) \ \simeq D^{\mu(t)}(A^T\Phi_M(t)\,) = \ A^T D^{\mu(t)}\Phi_M(t) = A^T\Theta G(t)\,\Theta^{-1}\Phi_M(t). \tag{26}$$

Secondly, $D^{\zeta_i(t)}\Phi_M(t)$, $i=1,2,3,4,\cdots n$. can be obtained by follow the same way that proposed for obtaining the $(NBOM)$ of $D^{\mu(t)}\Phi_M(t)$, as follows:

$$D^{\zeta_i(t)}\Phi_M(t) \ = \ (\Theta Q_i(t)\,\Theta^{-1})\Phi_M(t)\,,\ i=1,2,3,4,\cdots n. \tag{27}$$

Where

$$Q_i(t)=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \dfrac{\Gamma(2)t^{-\zeta_i(t)}}{\Gamma(2-\zeta_i(t))} & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{\Gamma(3)t^{-\zeta_i(t)}}{\Gamma(3-\zeta_i(t))} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\Gamma(1+M)t^{-\zeta_i(t)}}{\Gamma(1+M-\zeta_i(t))} \end{bmatrix}. \tag{28}$$

By using Eqs.(26) and (27), hence the Eq.(1) turned into

$$\begin{gathered}(A^T\Theta G(t)\,\Theta^{-1}\Phi_M(t)) \ = \\ F\,(t,\ A^T\Phi_M(t),\ (A^T\Theta Q_1(t)\,\Theta^{-1}\Phi_M(t)),\ (A^T\Theta Q_2(t)\,\Theta^{-1}\Phi_M(t)),\ ..., \\ (A^T\Theta Q_n(t)\,\Theta^{-1}\Phi_M(t)))\,,\ 0\le t\le T\,,\end{gathered} \tag{29}$$

with condition

$$A^T\Phi_M(0)=z_0.$$

Ultimately, we utilize t_k ($k = 0, 1, 2, 3, \cdots, M.$) where selected as $t_k = \dfrac{T(2k+1)}{2M+1}$(collocation point). Consequently Eq. (29) converted into the following form

$$\begin{aligned}
&A^T \Theta G(t_k)\, \Theta^{-1} \Phi_M(t_k) = \\
&F(t_k, A^T\Phi_M(t_k), (A^T\Theta Q_1(t_k)\Theta^{-1}\Phi_M(t_k)), (A^T\Theta Q_2(t_k)\Theta^{-1}\Phi_M(t_k)), ..., (A^T\Theta Q_n(t_k)\Theta^{-1}\Phi_M(t_k))), \\
&\qquad\qquad k = 0, 1, 2, 3, 4, \cdots, M. \qquad (30)
\end{aligned}$$

We can numerically solve the algebraic system in Eq.(30) to identify the matrix A. Thus, the numerical solution provided in Eq.(16) can be achieved.

5 Numerical experiences

Through demonstrating various examples in this section, we illustrate the effectiveness of the method utilizing the Mathematica 13 software.

To evaluate our approach, we implement our technique on several nonlinear multi-term variable-order FDEs. These examples have been examined in terms of the maximum Absolute Errors ($E_{max} = \max \mid Z_{Exact}(t_i) - Z_M(t_i) \mid, i = 0, 1, 2, 3, \cdots, M$), the Absolute Errors ($E_A = \mid Z_{Exact}(t_i) - Z_M(t_i) \mid$) and the time needed to compute their solution.

Collation of the outcomes produced by this method with the precise solution and with different other techniques (found in the literature) for each example indicates that this innovative technique aligns most closely with the exact solution. The consistency, stability, and straightforward application of this technique make it more usable and dependable.

Example 5.1. [32]: Examine the subsequent multi-term $VFDE$

$$\begin{gathered}
D^{\mu(t)}z(t) + t^{\frac{1}{2}}D^{\zeta_1(t)}z(t) + t^{\frac{1}{3}}D^{\zeta_2(t)}z(t) + t^{\frac{1}{4}}D^{\zeta_3(t)}z(t) + t^{\frac{1}{5}}z(t) = \\
-\frac{t^{2-\mu(t)}}{\Gamma(3-\mu(t))} - t^{\frac{1}{2}}\frac{t^{2-\zeta_1(t)}}{\Gamma(3-\zeta_1(t))} - t^{\frac{1}{3}}\frac{t^{2-\zeta_2(t)}}{\Gamma(3-\zeta_2(t))} - t^{\frac{1}{4}}\frac{t^{2-\zeta_3(t)}}{\Gamma(3-\zeta_3(t))} + t^{\frac{1}{5}}(\frac{4-t^2}{2}), \\
z(0) = 2,\ z^{'}(0) = 0,\ 0 \le t \le T.
\end{gathered} \tag{31}$$

Note that $z(t) = (\dfrac{4-t^2}{2})$ is the exact solution and $\mu(t) = 2t, \zeta_1(t) = \dfrac{t}{3}, \zeta_2(t) = \dfrac{t}{4}, \zeta_3(t) = \dfrac{t}{5}$.

Utilizing the ideas outlined in Section 4, we regard the approximate solution featuring $(M+1)$ finite terms shown in Eq.(16) for this problem and replace it in the primary problem. Then, by applying the Eqs.(26) and (27), this problem is transformed into the form of Eq.(29). Finally, utilizing t_j, we derive a system of equations that can be solved using Newton's iteration method to obtain the unknown matrix A.

The maximum absolute errors of this approach are presented in Table (1). From this table, it can be seen that the numerical outcomes are nearly identical to the exact solution, and Figure (1) confirms the reliability of the $NBOM$ method compared to other techniques. It is important to observe that Figure (1) illustrates a good alignment between the approximate solutions and the true solution.

Table 1: Comparison of E_{max} between method given in [32] and proposed technique for Example 5.1 with disjoint sets of T and M.

	Method in [32]			Our proposed method		
T	$M=2$	$M=3$	$M=4$	$M=2$	$M=3$	$M=4$
1	0	2.2204×10^{-16}	2.2204×10^{-16}	0	0	0
2	0	4.4409×10^{-16}	1.3323×10^{-15}	0	0	0
4	2.2204×10^{-16}	3.5527×10^{-15}	3.1974×10^{-14}	0	0	0

In this instance for $M=2$, we have

$$A\ =[1.83333,-0.5,-0.5]^T,$$

Then

$$z_2(t)=a_0B_0(t)+a_1B_1(t)+a_2B_2(t)=(1.83333\times1)+(-0.5\times(t-\frac{1}{2}))+(-0.5\times(t^2-t+\frac{1}{6}))=\frac{4-t^2}{2} \tag{32}$$

.

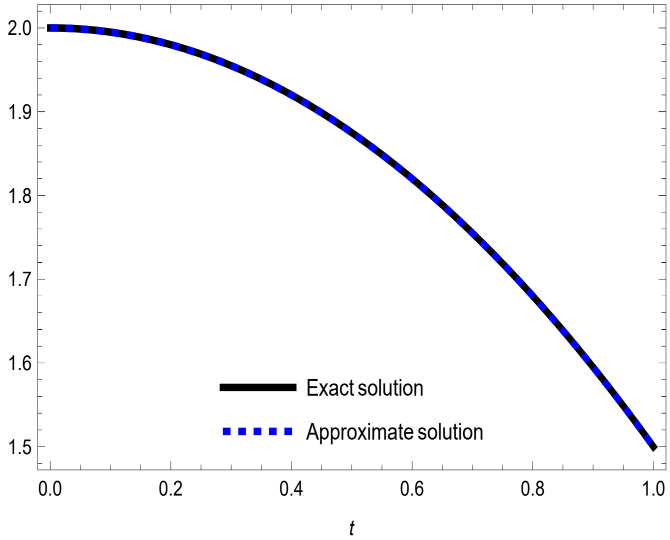

Figure 1: Comparison of between approximate solution(z_2) of $NBOM$ method and exact solution for Example 5.1.

Example 5.2. Assume the following equation:

$$D^{\mu(t)}z(t)+2D^{\zeta_1(t)}z(t)\ +4z(t)\ =\frac{\Gamma(3)}{\Gamma(3-\mu(t))}t^{2-\mu(t)}+\frac{2\Gamma(3)}{\Gamma(3-\zeta_1(t))}t^{2-\zeta_1(t)}+4t^2,$$
$$z(0)=0,\ \ z^{'}(0)=0,\ \ 0\leq t\leq T. \tag{33}$$

Note that $z(t) = t^2$ is the exact solution and $\mu(t) = 2t, \zeta_1(t) = \dfrac{1+t}{2}$.

Like the previous example, in this instance, for $M = 2$, we have

$$A = [0.3333333, 1, 1]^T,$$

So

$$z_2(t) = a_0B_0(t) + a_1B_1(t) + a_2B_2(t) = (0.3333333 \times 1) + (1 \times (t - \frac{1}{2})) + (1 \times (t^2 - t + \frac{1}{6})) = t^2 \tag{34}$$

Which is in full accordance with the exact solution. Also, Figure (2) supports the results, and the CPU time needed for our method related to this example is 0.0156 seconds.

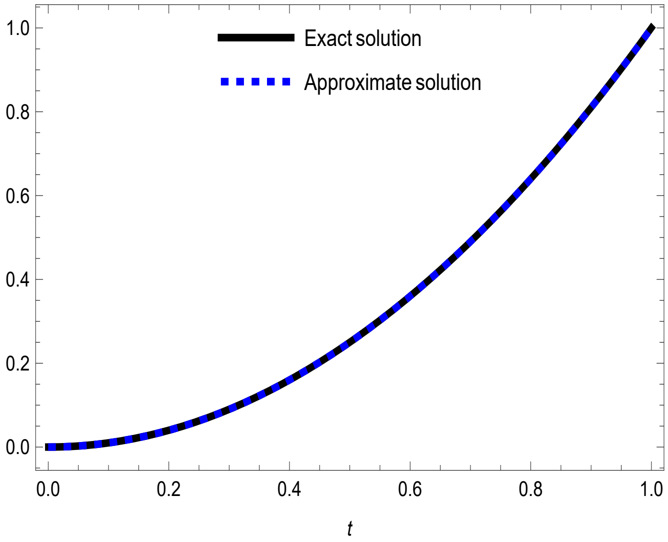

Figure 2: Comparison of between approximate solution(z_2) of $NBOM$ method and exact solution for Example 5.2.

Example 5.3. [48]: (The damped mechanical oscillator) Let us now consider the below equation:

$$\begin{gathered} D^{\mu(t)}z(t) + 2D^{\zeta_1(t)}z(t) + 4z(t) = 2 + 4t + 4t^2, \\ z(0) = 0, \ z'(0) = 0, \ 0 \le t \le T. \end{gathered} \tag{35}$$

Note that $z(t) = t^2$ is the exact solution and $\mu(t) = 2, \zeta_1(t) = 1$.

After solve this example similary to the previous examples and obtain results, for $M = 2$, we have

$$A = [0.3333333, 1, 1]^T,$$

Then

$$z_2(t) = a_0B_0(t) + a_1B_1(t) + a_2B_2(t) = (0.3333333 \times 1) + (1 \times (t - \frac{1}{2})) + (1 \times (t^2 - t + \frac{1}{6})) = t^2 \tag{36}$$

Table 2: Comparison the absolute errors at some nodal points for Example 5.4.

t_i	Our technique,$M = 2$
0.1	5.55×10^{-17}
0.2	5.37×10^{-17}
0.3	4.85×10^{-17}
0.4	4.16×10^{-17}
0.5	2.77×10^{-17}
1.0	0
0.6	0
0.7	0
0.8	1.11×10^{-16}
0.9	0
1.0	0
CPU time	$0.006\ s$

Which is the exact solution.

Example 5.4. [49]: (Bagley-Torvik equation) Consider the below multi-term fractional variable-order differential equation:

$$\begin{aligned} D^{\mu(t)}z(t) + D^{\zeta_1(t)}z(t)\ + z(t)\ &= 2 + t^2 + \frac{4}{\sqrt{\pi}}t^{0.5}, \\ z(0) = 0,\ z^{'}(0) &= 0,\ 0 \leq t \leq T. \end{aligned} \tag{37}$$

Note that $z(t) =\ t^2$ is the exact solution and $\mu(t) = 2, \zeta_1(t) = \dfrac{3}{2}$.

We solve this example (Eq. (37)) similary to the previous examples and obtain results, Then set up the table of Absolute Errors (at some nodal points) and CPU time needed for our method related to it (Table(2)) and also draw the Figure (3) to demonstrate its accuracy and efficiency.

In this instance, similarly for $M = 2$, we have

$$A\ = [0.3333333, 1, 1]^T, z_2(t) = t^2$$

Which is in full accordance with the exact solution.

Example 5.5. Consider the following nonlinear $VFDE$ taken from

$$D^{\mu(t)}z(t) + \sin(t)z^2(t) = \frac{\Gamma(\frac{9}{2})}{\Gamma(\frac{9}{2} - \mu(t))}t^{\frac{7}{2}-\mu(t)} + \sin(t)t^7, \quad z(0) = 0,\ 0 \leq t \leq T. \tag{38}$$

Note that $z(t) =\ t^{\frac{7}{2}}$ is the exact solution and $\mu(t) = 1 - 0.5\exp(-t)$.

Like the previous examples, we solve this problem with different values of M. The numerical outcomes are displayed in Table(3) and Figure (4). In Figure (4), the approximate solutions obtained

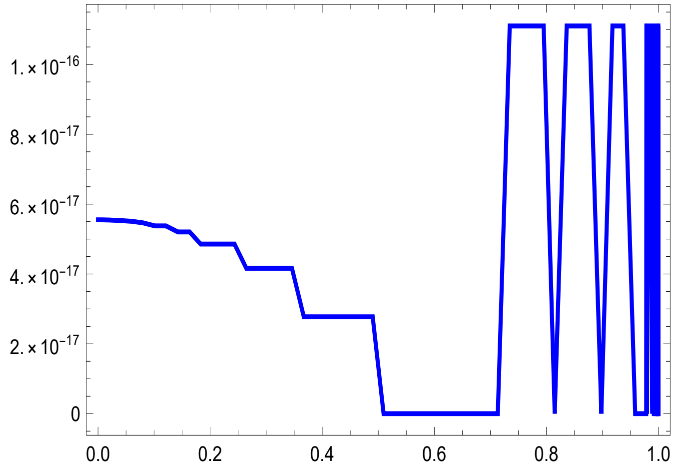

Figure 3: The absolute errors comparison between numerical solution(z_2) of $NBOM$ scheme and exact solution for Example 5.4.

with $M = 1, 2, 3, 4$, together with the exact solution of this problem, are plotted. The Absolute Errors (at specific nodal points) of this approach, along with the required CPU time, are presented in Table (3). From this Table, it can be seen that the numerical outcomes are nearly identical to the exact solution, and Figures (4) and (5) confirm the reliability of $NBOM$ method compared to other techniques. Furthermore, From these results, the convergence of the numerical solutions to the exact one can be easily seen.

6 Conclusion

In this paper, the fundamental objective of the paper is to introduce a novel computational methods based on Bernoulli operational matrix ($NBOM$) for the generalized non-linear multi-term variable-order FDEs Eqs. (1)-(2). We investigated the generalized $VFDE$s by the ($NBOM$) scheme. In this method, an algebraic equations system is obtained and solved by using a appropriate numerical method. The precision and effectiveness of this method is shown by solving some numerical examples.

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Table 3: Absolute errors at some selected points with different M for Example 5.5.

Our proposed method

t_i	$M=3$	$M=6$	$M=10$	$M=15$
0	0	3.59×10^{-16}	1.78×10^{-13}	4.20×10^{-10}
0.1	1.90×10^{-3}	8.06×10^{-6}	1.64×10^{-7}	4.82×10^{-8}
0.2	5.51×10^{-4}	1.64×10^{-6}	2.63×10^{-7}	$4.93 \times 10^{-8}0$
0.3	1.07×10^{-3}	2.84×10^{-6}	2.04×10^{-7}	4.51×10^{-8}
0.4	1.47×10^{-3}	3.32×10^{-6}	2.02×10^{-7}	3.68×10^{-8}
0.5	4.07×10^{-3}	7.80×10^{-6}	1.81×10^{-7}	2.59×10^{-8}
0.6	1.35×10^{-3}	2.32×10^{-6}	1.80×10^{-7}	1.55×10^{-8}
0.7	2.10×10^{-3}	3.58×10^{-6}	1.50×10^{-7}	8.93×10^{-9}
0.8	7.45×10^{-4}	3.78×10^{-6}	1.83×10^{-7}	8.33×10^{-9}
0.9	1.05×10^{-2}	1.02×10^{-5}	1.01×10^{-7}	1.61×10^{-8}
1.0	3.13×10^{-2}	1.98×10^{-4}	1.15×10^{-5}	1.41×10^{-6}
$CPUtime$	0.045s	0.125s	0.48s	1.592s

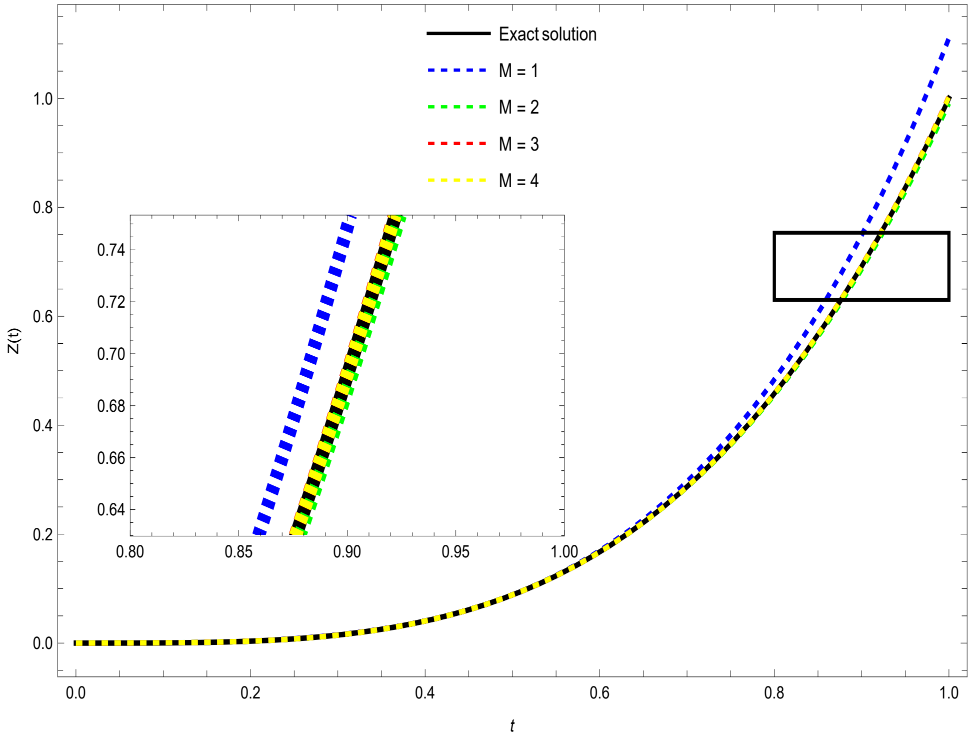

Figure 4: Comparison between the exact solution and approximate solution (z_M) of the $NBOM$ method with $M = 1, 2, 3, 4$ for Example 5.5.

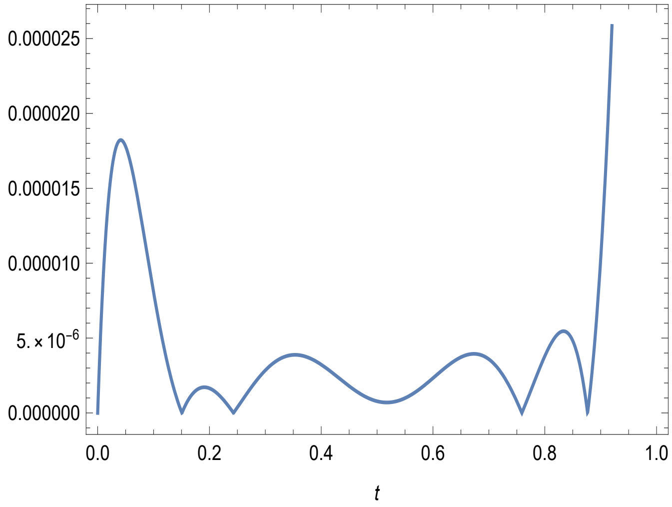

Figure 5: The absolute errors comparison between numerical solution(z_6) of $NBOM$ scheme and exact solution for Example 5.5.

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