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Research Article

Necessary Optimality Conditions for Weakly Efficient Solutions in Convex Multiobjective GSIPs

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Abstract

Abstract: In this paper, we investigate multiobjective generalized semi-infinite optimization problems with nondifferentiable convex data. We introduce several upper-level qualification conditions of Cottle-type for both the constraints and the data. Based on these qualification conditions, we establish first-order necessary optimality conditions of Fritz–John, Karush–Kuhn–Tucker, and strong Karush–Kuhn–Tucker types for weakly efficient solutions of the considered problems. The results are derived using tools from convex analysis.
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1 Introduction

Multiobjective semi-infinite optimization problems (MSIPs) arise in various scientific, engineering, and economic contexts where multiple conflicting objectives must be optimized simultaneously under infinitely many inequality constraints. A general MSIP can be formulated as

$$\min \ \big(f_1(x), f_2(x), \ldots, f_p(x)\big) \quad \text{s.t.} \quad g_j(x) \geq 0, \quad \forall j \in J,$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in \{1, \ldots, p\}$ are the objective functions, $g_j : \mathbb{R}^n \to \mathbb{R}$ for $j \in J$ are the constraint functions, and J is an arbitrary index set, not necessarily finite or compact. Necessary optimality conditions for linear, convex, and non-convex MSIPs have been investigated in several works; see, e.g., [2, 3, 10, 12, 14] and the references therein.

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A *multiobjective generalized semi-infinite program* (MGSIP) extends the classical MSIP by allowing infinitely many constraints whose index set depends explicitly on the decision variables:

$$\begin{aligned} \min \quad & F(x) := \big(f_1(x), f_2(x), \dots, f_p(x)\big) \\ \text{s.t.} \quad & g(x,y) \geq 0, \quad \forall y \in Y(x), \end{aligned}$$

where $F : \mathbb{R}^n \to \mathbb{R}^p$ denotes the vector-valued objective function, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the constraint function, and $Y(x) \subseteq \mathbb{R}^m$ is a nonempty index set that depends on the decision variable x.

In many practical applications, both F and g are nondifferentiable convex functions [20]. Nondifferentiability may arise from the presence of support functions, absolute values, piecewise-defined models, indicator functions, or Euclidean norms. The interplay of multiple objectives, variable-dependent infinite constraints, and convex analysis renders the study of *convex MGSIPs* both mathematically rich and practically significant.

In the present paper, motivated by [7, 8, 11, 13, 22, 23], we consider the following MGSIP:

$$(P): \quad \min_{x \in S} \big(f_1(x), \dots, f_p(x)\big),$$

with the feasible set

$$S := \{\, x \in \mathbb{R}^n \mid g(x,y) \geq 0,\ \forall y \in Y(x) \,\},$$

and the index set

$$Y(x) := \{\, y \in \mathbb{R}^m \mid h_t(x,y) \leq 0,\ \forall t \in T \,\},$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in I := \{1, \dots, p\}$, and $g, h_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ for $t \in T$ are convex functions. The index set T is finite, and the set-valued mapping $x \mapsto Y(x)$ is assumed to be uniformly bounded; that is, $Y(x)$ is compact-valued and upper semicontinuous at each $x_0 \in S$ (cf. [1]). These assumptions will be maintained throughout the sequel.

The aim is not to determine a single minimizer, but rather the *weakly efficient set*, consisting of points at which no objective can be improved simultaneously. A feasible point $x^* \in S$ is said to be a *weakly efficient solution* of (P) if there does not exist any $x \in S$ such that

$$f_i(x) < f_i(x^*), \quad \text{for all } i \in I.$$

When $p = 1$, the MGSIP reduces to the *generalized semi-infinite programming problem* (GSIP), a well-established and active research area in optimization theory. In the majority of the literature on GSIPs, the derivation of optimality conditions for problem (P) relies on various lower-level constraint qualifications (CQs). A comprehensive account of such CQs and the corresponding optimality conditions—together with their applications and historical development—in the setting where all functions are continuously differentiable (but not necessarily convex) is provided in the monograph by Stein [22]. Further extensions of these CQs and optimality conditions to GSIPs with convex functions, with locally Lipschitz data and with DC (difference-of-convex) structures have been investigated by Soroush [21], by Kanzi and Nobakhtian [13] and by Kanzi [8, 11], respectively. However, significantly fewer results are available for GSIPs and MGSIPs involving nondifferentiable convex data, leaving an important gap that motivates the present study.

When all functions in the MGSIP are continuously differentiable (resp. convex, and locally Lipschitz), certain first-order necessary conditions have been established in [23] (resp. [9], and [6]). In particular, Kanzi *et al.* [9] employ Abadie- and Guignard-type constraint qualifications to derive necessary optimality conditions for weakly efficient solutions of convex MGSIPs. However, the justification of these CQs depends on the computation of *tangent cones*, which is technically demanding in variational analysis. This highlights a methodological gap: although such conditions are theoretically sound, their practical verification is often cumbersome.

In this paper, we address this gap by introducing several Cottle-type qualification conditions, formulated both at the constraint and data levels. Unlike Abadie- and Guignard-type CQs, the verification of Cottle-type conditions reduces to elementary algebraic arguments, making them substantially more tractable. Building on these qualifications, we establish Fritz–John, Karush–Kuhn–Tucker, and strong Karush–Kuhn–Tucker type first-order necessary optimality conditions for weakly efficient solutions of convex MGSIPs.

The remainder of this paper is organized as follows. Section 2 presents the necessary definitions and preliminary results from convex analysis that will be used throughout the paper. Section 3, which constitutes the main contribution, introduces several qualification conditions and establishes a number of necessary optimality conditions for nondifferentiable convex MGSIPs.

2 Preliminaries

In this section, we provide a concise overview of basic definitions and standard preliminaries from convex analysis that will be used in the sequel; see [5, 18] for details.

We denote by $\mathbb{R}_{++}$ and $\mathbb{R}_+$ the sets of strictly positive and nonnegative real numbers, respectively, i.e.,

$$\mathbb{R}_{++} := (0, +\infty) \quad \text{and} \quad \mathbb{R}_+ := \overline{\mathbb{R}_{++}} = [0, +\infty),$$

where $\overline{A}$ denotes the closure of a set $A \subseteq \mathbb{R}^n$. The standard inner product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y\rangle$, and the zero vector in $\mathbb{R}^n$ is denoted by 0_n.

Given a nonempty set $A \subseteq \mathbb{R}^n$, the notations $conv(A)$ and $\overline{conv}(A)$ denote the convex hull and the closed convex hull of A, respectively. Furthermore, the strictly positive polar and strictly negative polar of a set A are defined, respectively, as

$$A^{>} := \big\{x \in \mathbb{R}^n \mid \langle x, a\rangle > 0, \quad \forall a \in A\big\},$$
$$A^{<} := \big\{x \in \mathbb{R}^n \mid \langle x, a\rangle < 0, \quad \forall a \in A\big\} = -A^{>}.$$

Recall that if $A = \emptyset$, then by definition, $A^{>} = A^{<} = \mathbb{R}^n$.

Theorem 2.1. *Let $A \subseteq \mathbb{R}^n$ be a compact also set. Then,*

- *$conv(A)$ is compact ([5, Theorem 1.4.3]).*
- *$0_n \notin conv(A)$ if and only if $A^{>} \neq \emptyset$ ([4, Theorem 3.2]).*

We note [18, Theorem 6.9] that if $\Pi := \{B_\ell \mid \ell \in L\}$ is a collection of convex sets in $\mathbb{R}^n$, then

$$conv\Big(\bigcup_{\ell\in L} B_\ell\Big) = \bigcup_{\{B_{\ell_1},\dots,B_{\ell_{n+1}}\}\subseteq\Pi} \left\{\sum_{\nu=1}^{n} \lambda_\nu B_{\ell_\nu} \;\middle|\; \lambda_\nu \geq 0,\ \sum_{\nu=1}^{n+1} \lambda_\nu = 1\right\}. \tag{1}$$

A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be *convex* if, for all $x, y \in \mathbb{R}^n$,

$$\varphi\big(\lambda x + (1-\lambda)y\big) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y), \qquad \forall \lambda \in [0, 1].$$

The *subdifferential* of a convex function φ at $x_0 \in \mathbb{R}^n$ is defined as

$$\partial\varphi(x_0) := \{\xi \in \mathbb{R}^n \mid \varphi(x) - \varphi(x_0) \geq \langle \xi, x - x_0 \rangle, \ \forall x \in \mathbb{R}^n\}.$$

We recall from [5], if φ is convex, its classical directional derivative $\varphi'(x_0; d)$, defined by

$$\varphi'(x_0; d) := \lim_{\varepsilon \to 0^+} \frac{\varphi(x_0 + \varepsilon d) - \varphi(x_0)}{\varepsilon},$$

exists, and we have

$$\partial\varphi(x_0) = \{\xi \in \mathbb{R}^n \mid \varphi'(x_0; d) \geq \langle \xi, d \rangle, \ \forall d \in \mathbb{R}^n\}.$$

Since $\partial\varphi(x_0)$ is always a non-empty, compact, and convex set in $\mathbb{R}^n$ [5], the above equality implies that

$$\varphi'(x_0; d) = \max\{\langle \xi, d \rangle \mid \xi \in \partial\varphi(x_0)\}. \tag{2}$$

Note that if φ is differentiable at x_0, then $\partial\varphi(x_0) = \{\nabla\varphi(x_0)\}$, where $\nabla\varphi(x_0)$ denotes the gradient of φ at x_0.

Assume that $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a convex function, and let $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$. Then, the functions $\psi(\cdot, y_0) : \mathbb{R}^n \to \mathbb{R}$ and $\psi(x_0, \cdot) : \mathbb{R}^m \to \mathbb{R}$ are both convex. The *partial subdifferentials* of ψ at (x_0, y_0) are denoted by $\partial_x\psi(x_0, y_0) \subseteq \mathbb{R}^n$ and $\partial_y\psi(x_0, y_0) \subseteq \mathbb{R}^m$, and are defined as

$$\partial_x\psi(x_0, y_0) := \partial\psi(\cdot, y_0)(x_0), \quad \text{and} \quad \partial_y\psi(x_0, y_0) := \partial\psi(x_0, \cdot)(y_0).$$

3 Necessary Conditions

At the outset of this section, we introduce some notations.

For each $x_0 \in S$, we define the index set of active constraints and the corresponding lower-level problem at x_0 as

$$Y_0(x_0) := \{y \in Y(x_0) \mid g(x_0, y) = 0\},$$

$$(LP_{x_0}) : \quad \min \ g(x_0, y), \quad \text{s.t. } y \in Y(x_0).$$

Moreover, the set of active inequalities of (LP_{x_0}) at each $y_0 \in Y(x_0)$, which may be empty, is denoted by

$$T_0(x_0, y_0) := \{t \in T \mid h_t(x_0, y_0) = 0\}.$$

If $y_0 \in Y_0(x_0)$, the Fritz–John (FJ) multiplier set of (LP_{x_0}) at y_0 is denoted by $F(x_0, y_0)$ and consists of all $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+^{|T_0(x_0,y_0)|}$ satisfying

$$\begin{cases} \alpha + \displaystyle\sum_{t \in T_0(x_0,y_0)} \beta_t = 1, \\ 0_m \in \alpha \, \partial_y g(x_0, y_0) + \displaystyle\sum_{t \in T_0(x_0,y_0)} \beta_t \, \partial_y h_t(x_0, y_0). \end{cases} \tag{3}$$

Equivalently,

$$F(x_0, y_0) := \left\{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+^{|T_0(x_0,y_0)|} \;\middle|\; (3) \text{ holds}\right\}.$$

Let $x_0 \in S$ be a feasible point for (P). We define

$$\Omega(x_0) := \bigcup_{y \in Y_0(x_0)} \left\{ \alpha\, \partial_x g(x_0, y) + \sum_{t \in T_0(x_0,y)} \beta_t\, \partial_x h_t(x_0, y) \;\middle|\; (\alpha, \beta) \in F(x_0, y) \right\}.$$

We note that $\Omega(x_0)$ serves as the nonsmooth counterpart of $V(x_0)$ defined in [7, 23].

The following theorem is a natural generalization of [7, Lemma 3.3] and plays an important role in the subsequent analysis.

Theorem 3.1. *Assume that a feasible point $\hat{x} \in S$ is given. Then, $\Omega(\hat{x})$ is a compact set.*

Proof. Suppose that $\{\xi^k\}_{k=1}^{\infty}$ is a sequence in $\Omega(\hat{x})$. Thus, we can find some sequences $\{y^k\} \subseteq Y_0(\hat{x})$ and $\{(\alpha^k, \beta^k)\} \subseteq F(\hat{x}, y^k)$ such that

$$\xi^k \in \alpha^k \partial_x g(\hat{x}, y^k) + \sum_{t \in T_0(\hat{x}, y^k)} \beta_t^k \partial_x h_t(\hat{x}, y^k), \qquad \text{for all } k \in \mathbb{N}.$$

Note that

$$0_m \in \alpha^k \partial_y g(\hat{x}, y^k) + \sum_{t \in T_0(\hat{x}, y^k)} \beta_t^k \partial_y h_t(\hat{x}, y^k), \qquad \text{for all } k \in \mathbb{N}. \tag{4}$$

Define $\beta^k := \left(\beta_1^k, \ldots, \beta_\rho^k\right)$ and

$$T_k := T_0(\hat{x}, y^k), \qquad \text{for all } k \in \mathbb{N}.$$

Since ,for each $k \in \mathbb{N}$, we have $|T_k| \leq \rho$ and the number of k is infinity, some of T_ks repeat infinite times. Let $\widetilde{T}$ be one of these repeated sets, i.e., after a relabeling,

$$\widetilde{T} := T_0(\hat{x}, y^k), \qquad \text{for all } k \in \mathbb{N}.$$

Note that the compactness of $Y_0(\hat{x})$ and the the fact that

$$(\alpha^k, \beta^k) \in [0,1] \times [0,1]^{|\widetilde{T}|}, \qquad \text{for all } k \in \mathbb{N},$$

imply that, with no relabeling of subsequences,

$$y^k \to \hat{y} \in Y_0(\hat{x}) \quad \text{and} \quad (\alpha^k, \beta^k) \to (\hat{\alpha}, \hat{\beta}),$$

such that

$$\hat{\alpha} + \sum_{t \in \widetilde{T}} \hat{\beta}_t = 1. \tag{5}$$

On the other hand, the continuity of h_t functions as $t \in T$ concludes that

$$\overbrace{h_t(\hat{x}, y^k)}^{=0} \to h_t(\hat{x}, \hat{y}), \qquad \text{for all } t \in \widetilde{T},$$

and so $h_t(\hat{x},\hat{y})=0$ as $t\in\widetilde{T}$, which deduces that $\widetilde{T}\subseteq T_0(\hat{x},\hat{y})$. Hence, the upper semicontinuity of convex subdifferential and (4) conclude that, with no relabeling of subsequences,

$$0_m\in\hat{\alpha}\partial_y g(\hat{x},\hat{y})+\sum_{t\in\tilde{T}}\hat{\beta}_t\partial_y h_t(\hat{x},\hat{y})\subseteq\hat{\alpha}\partial_y g(\hat{x},\hat{y})+\sum_{t\in T_0(\hat{x},\hat{y})}\hat{\beta}_t\partial_y h_t(\hat{x},\hat{y}).$$

This inclusion and (5) show that

$$(\hat{\alpha},\hat{\beta})\in F(\hat{x},\hat{y}).\tag{6}$$

Now, the upper semicontinuity of set-valued mapping $y\mapsto\partial_x g(\hat{x},y)$ and $y\mapsto\partial_x h_t(\hat{x},y)$ as $t\in T$ conclude that we can find a subsequence of $\{\xi^k\}$ converging to some $\hat{\xi}$ with

$$\hat{\xi}\in\hat{\alpha}\partial_x g(\hat{x},\hat{y})+\sum_{t\in T_0(\hat{x},\hat{y})}\hat{\beta}_t\partial_x h_t(\hat{x},\hat{y})\subseteq\Omega(\hat{x}),$$

where the final inclusion holds by (6). The proof is complete. ■

As we noted in Section 1, if $Y(x):=Y$ for all $x\in S$, the problem (P) can be reformulated as the following MSIP:

$$\begin{array}{llll}(P_1): & \min & (f_1(x),\dots,f_p(x)) & \\ & \text{s.t.} & g_y(x)\geq 0, & y\in Y,\end{array}$$

where the convex function $g_y:\mathbb{R}^n\to\mathbb{R}$, for each $y\in Y$, is defined as

$$g_y(x):=g(x,y),\quad\forall x\in\mathbb{R}^n.$$

To establish the necessary optimality conditions for (P_1), the following two assumptions are commonly made and have been used in numerous studies; see, e.g., [15] for the convex case, and [2, 10, 12] for the nonsmooth case:

- The Pshenichnyi-Levin-Valadier (PLV) property at x_0, i.e.,

  $$\partial\big(\inf_{y\in Y}g_y\big)(x_0)\subseteq conv\Big(\bigcup_{y\in Y_{x_0}}\partial g_y(x_0)\Big),$$

  where

  $$Y_{x_0}:=\big\{y\in Y\mid g_y(x_0)=0\big\}.$$

- The continuity property at x_0, i.e., Y is a compact set (in some Euclidean spaces), the function $y\to g_y(x_0)$ is upper semicontinuous on Y, and the set-valued function $y\mapsto\partial g_y(x_0)$ is upper semicontinuous on Y.

To extend these conditions to MGSIPs, we consider the following assumptions, while the remaining conditions are derived from our previous hypotheses.

Definition 3.2. We say that the *perfect property* (PP) holds at $x_0\in S$ if the following conditions are satisfied:

- The extended PLV property holds at x_0, i.e.,

$$\partial\varphi(x_0) \subseteq conv\big(\Omega(x_0)\big),$$

where

$$\varphi(x) := \inf\{g(x,y) \mid y \in Y(x)\}, \qquad \forall x \in S.$$

- The set-valued mappings $y \mapsto \partial_x g(x_0, y)$ and $y \mapsto \partial_x h_t(x_0, y)$, for all $t \in T$, are upper semicontinuous on $Y(x_0)$.

It is worth noting that, because of the significance of the function $\varphi(\cdot)$, known as the "marginal function," numerous studies have focused on obtaining upper estimates of its subdifferential; see, e.g., [16, 17, 19] and the references therein.

Remark 3.3. Note that if the functions involved are continuously differentiable, the perfect property (PP) automatically holds at every $x_0 \in S$.

The following lemma plays a fundamental role in the subsequent analysis.

Lemma 3.4. *Suppose that $\hat{x}$ is a weakly efficient solution of (P) and that the PP holds at $\hat{x}$. Then,*

$$0_n \in conv\Bigg(-\bigcup_{i=1}^{p} \partial f_i(\hat{x}) \;\cup\; \Omega(\hat{x})\Bigg). \tag{7}$$

Proof. Since $\bigcup_{i=1}^{p} \partial f_i(\hat{x})$ is a finite union of compact sets, it is compact, and Theorem 3.1 concludes that the following set is compact,

$$-\bigcup_{i=1}^{p} \partial f_i(\hat{x}) \cup \Omega(\hat{x}).$$

If (7) does not hold, Theorem 2.1 implies that

$$\Bigg(-\bigcup_{i=1}^{p} \partial f_i(\hat{x}) \cup \Omega(\hat{x})\Bigg)^{>} \neq \emptyset.$$

Thus, we can find a non-zero vector $u \in \mathbb{R}^n$ such that

$$0_n \neq u \in \Bigg(-\bigcup_{i=1}^{p} \partial f_i(\hat{x}) \cup \Omega(\hat{x})\Bigg)^{>} = \Bigg(\bigcup_{i=1}^{p} \partial f_i(\hat{x})\Bigg)^{<} \cap \big(\Omega(\hat{x})\big)^{>}.$$

Since

$$u \in \big(\Omega(\hat{x})\big)^{>} = \Big(conv\big(\Omega(\hat{x})\big)\Big)^{>},$$

the PP assumption implies that $u \in \big(\partial\varphi(\hat{x})\big)^{>}$. So, $\langle u, \xi\rangle > 0$ for all $\xi \in \partial\varphi(\hat{x})$, which concludes $\varphi'(\hat{x}; u) > 0$ by (2). Thus, by the definition of directional derivative, we can find a positive number $\delta_0 > 0$ such that

$$\varphi(\hat{x} + \varepsilon u) - \varphi(\hat{x}) > 0, \quad \text{for all } \varepsilon \in (0, \delta_0).$$

This inequality, the fact that $\varphi(\hat{x}) \geq 0$, and definition of $\varphi(\cdot)$, conclude that

$$g(\hat{x} + \varepsilon u, y) > 0, \qquad \forall \varepsilon \in (0, \delta_0),\ \forall y \in Y(\hat{x} + \varepsilon u). \tag{8}$$

On the other hand, because

$$u \in \left(\bigcup_{i=1}^{p} \partial f_i(\hat{x}) \right)^{<} = \bigcap_{i=1}^{p} \left(\partial f_i(\hat{x}) \right)^{<},$$

we deduce by (2) that

$$f_i'(\hat{x}; u) < 0, \quad \text{for all } i \in I.$$

Thus, for each $i \in I$, we can find a positive number δ_i such that

$$f_i(\hat{x} + \varepsilon u) < f_i(\hat{x}), \qquad \text{for all } \ \varepsilon \in (0, \delta_i). \tag{9}$$

Take

$$\delta := \min \left\{ \delta_0, \delta_1, \ldots, \delta_p \right\}.$$

Owing to (8)-(9), for all $\varepsilon \in (0, \hat{\delta})$, we have $\hat{x} + \varepsilon u \in S$ and

$$\Big(f_1(\hat{x} + \varepsilon u), \ldots, f_p(\hat{x} + \varepsilon u) \Big) < \Big(f_1(\hat{x}), \ldots, f_p(\hat{x}) \Big).$$

This contradicts the weak efficiency of $\hat{x}$, and so (2) is proved. ■

The following theorem extends [7, Theorem 1.1] to the setting of nondifferentiable convex MGSIPs.

Theorem 3.5 (FJ Necessary Condition)**.** *Suppose that $\hat{x}$ is a weakly efficient solution of (P) and that the PP condition holds at $\hat{x}$. Then, there exist finitely many indices $y^1, \ldots, y^q \in Y_0(\hat{x})$, non-negative scalars $\eta^\nu \in \mathbb{R}_+$ and $\tau_t^\nu \in \mathbb{R}_+$ for $\nu = 1, \ldots, q$ and $t \in T_0(\hat{x}, y^\nu)$, as well as non-negative scalars $\lambda_i \in \mathbb{R}_+$ for $i \in I$, such that*

$$0_n \in \sum_{i=1}^{p} \lambda_i \partial f_i(\hat{x}) - \sum_{\nu=1}^{q} \eta^\nu \partial_x g(\hat{x}, y^\nu) - \sum_{\nu=1}^{q} \sum_{t \in T_0(\hat{x}, y^\nu)} \tau_t^\nu \partial_x h_t(\hat{x}, y^\nu), \tag{10}$$

$$\sum_{i=1}^{p} \lambda_i + \sum_{\nu=1}^{q} \eta^\nu + \sum_{\nu=1}^{q} \sum_{t \in T_0(\hat{x}, y^\nu)} \tau_t^\nu = 1. \tag{11}$$

Proof. According to Lemma 3.4 and the structure of convex hulls in (1), there exist points $y^\nu \in Y_0(\hat{x})$ and pairs $(\alpha^\nu, \beta^\nu) \in F(\hat{x}, y^\nu)$ for $\nu = 1, \ldots, q$, non-negative scalars $\lambda_i \in \mathbb{R}_+$ for $i \in I$, and non-negative scalars $\gamma_\nu \in \mathbb{R}_+$ for $\nu = 1, \ldots, q$, such that

$$\begin{aligned} 0_n \ &\in \ \sum_{i=1}^{p} \lambda_i \partial f_i(\hat{x}) - \left(\sum_{\nu=1}^{q} \gamma_\nu \Big(\alpha^\nu \partial_x g(\hat{x}, y^\nu) + \sum_{t \in T_0(\hat{x}, y^\nu)} \beta_t^\nu \partial_x h_t(\hat{x}, y^\nu) \Big) \right) \\ &= \ \sum_{i=1}^{p} \lambda_i \partial f_i(\hat{x}) - \sum_{\nu=1}^{q} \gamma_\nu \alpha^\nu \partial_x g(\hat{x}, y^\nu) - \sum_{\nu=1}^{q} \sum_{t \in T_0(\hat{x}, y^\nu)} \gamma_\nu \beta_t^\nu \partial_x h_t(\hat{x}, y^\nu), \end{aligned}$$

$$\sum_{i=1}^{p} \lambda_i + \sum_{\nu=1}^{q} \gamma_\nu = 1. \tag{12}$$

Taking

$$\eta^\nu := \gamma_\nu \alpha^\nu \ \text{ and } \ \tau_t^\nu := \gamma_\nu \beta_t^\nu, \ \text{ for all } \nu = 1, \ldots, q,$$

we have

$$0_n \in \sum_{i=1}^{p} \lambda_i \partial f_i(\hat{x}) - \sum_{\nu=1}^{q} \eta^\nu \partial_x g(\hat{x}, y^\nu) - \sum_{\nu=1}^{q} \sum_{t \in T_0(\hat{x}, y^\nu)} \tau_t^\nu \partial_x h_t(\hat{x}, y^\nu),$$

and

$$\sum_{i=1}^{p} \lambda_i + \sum_{\nu=1}^{q} \eta^\nu + \sum_{\nu=1}^{q} \sum_{t \in T_0(\hat{x}, y^\nu)} \tau_t^\nu = \sum_{i=1}^{p} \lambda_i + \sum_{\nu=1}^{q} \gamma_\nu \Big(\alpha^\nu + \sum_{t \in T_0(\hat{x}, y^\nu)} \beta_t^\nu \Big) = 1,$$

where the last equality follows from $(\alpha^\nu, \beta^\nu) \in F(\hat{x}, y^\nu)$ for $\nu = 1, \ldots, q$, and from (12). ■

As is well known, Theorem 3.5 remains valid even when $\lambda_i = 0$ for all $i \in I$, in which case the vector-valued objective function does not appear in the main inclusion (10). For this reason, we state a Karush–Kuhn–Tucker (KKT) type necessary optimality condition below. It is well established in classical optimization that a KKT-type necessary condition requires an appropriate constraint qualification (CQ). Motivated by this, we introduce the following two (upper-level) Cottle-type qualifications for problem (P).

Definition 3.6. We say that (P) satisfies

- the *Cottle constraint qualification*, denoted by CCQ, at $x_0 \in S$ if

$$\big(\Omega(x_0)\big)^{>} \neq \emptyset.$$

- the *Collte data qualification*, denoted by CDQ, at $x_0 \in S$ if

$$\Bigg(\bigcup_{i \in I \setminus k} \partial f_i(x_0) \Bigg)^{<} \cap \big(\Omega(x_0)\big)^{>} \neq \emptyset, \qquad \text{for all } k \in I.$$

Note that, since the objective functions play a role in the definition of CDQ, it is referred to as a *data qualification* rather than a *constraint qualification*. Clearly, the following implication holds at each feasible point $x_0 \in S$:

$$\text{CDQ} \implies \text{CCQ}. \tag{13}$$

As generalization of [23, Theorem 3.1], we set the following theorem under CCQ.

Theorem 3.7 (KKT Necessary Condition Under CCQ)**.** *Suppose that $\hat{x}$ is a weakly efficient solution of (P). If CCQ and PP hold at $\hat{x}$, then there exist finitely many indices $y^1, \ldots, y^q \in Y_0(\hat{x})$, non-negative scalars $\eta^\nu \in \mathbb{R}_+$ and $\tau_t^\nu \in \mathbb{R}_+$ for $\nu = 1, \ldots, q$ and $t \in T_0(\hat{x}, y^\nu)$, as well as non-negative scalars $\lambda_i \in \mathbb{R}_+$ for $i \in I$, such that*

$$0_n \in \sum_{i=1}^{p} \lambda_i \partial f_i(\hat{x}) - \sum_{\nu=1}^{q} \eta^\nu \partial_x g(\hat{x}, y^\nu) - \sum_{\nu=1}^{q} \sum_{t \in T_0(\hat{x}, y^\nu)} \tau_t^\nu \partial_x h_t(\hat{x}, y^\nu),$$

$$\sum_{i=1}^{p} \lambda_i = 1.$$

Proof. Employing Theorem 3.5, we can find points $y^1, \ldots, y^q \in Y_0(\hat{x})$, non-negative scalars $\eta^\nu \in \mathbb{R}_+$ and $\tau_t^\nu \in \mathbb{R}_+$ for $\nu = 1, \ldots, q$ and $t \in T_0(\hat{x}, y^\nu)$, as well as non-negative scalars $\lambda_i \in \mathbb{R}_+$ for $i \in I$, such that (10)–(11) are satisfied. If $\lambda_i = 0$ for all $i \in I$, we then obtain

$$0_n \in \sum_{\nu=1}^{q} \eta^\nu \partial_x g(\hat{x}, y^\nu) + \sum_{\nu=1}^{q} \sum_{t \in T_0(\hat{x}, y^\nu)} \tau_t^\nu \partial_x h_t(\hat{x}, y^\nu),$$

$$\sum_{\nu=1}^{q} \eta^\nu + \sum_{\nu=1}^{q} \sum_{t \in T_0(\hat{x}, y^\nu)} \tau_t^\nu = 1.$$

This implies $0_n \in conv\big(\Omega(\hat{x})\big)$, and so $\Big(conv\big(\Omega(\hat{x})\big)\Big)^> = \emptyset$. Consequently,

$$\big(\Omega(\hat{x})\big)^> = \Big(conv\big(\Omega(\hat{x})\big)\Big)^> = \emptyset,$$

which contradicts the CCQ assumption at $\hat{x}$. This contradiction implies that $\lambda_i \neq 0$ for at least one $i \in I$, and hence $\sum_{i=1}^{p} \lambda_i > 0$. Dividing all coefficients λ_i, η^ν, and τ_t^ν by $\sum_{i=1}^{p} \lambda_i$, and appropriately renaming the multipliers, yields the desired result. ■

In most examples, Theorem 3.7 involves $\lambda_k = 0$ for some $k \in I$, so that the corresponding objective function f_k does not contribute to the necessary condition. We say that the strong KKT condition holds at $\hat{x}$ when $\lambda_i > 0$ for all $i \in I$. In the next theorem, we state the strong KKT necessary conditions for a weakly efficient solution of (P) under the CDQ assumption.

Theorem 3.8 (Strong KKT Necessary Condition Under CDQ)**.** *Suppose that $\hat{x}$ is a weakly efficient solution of (P). If CDQ and PP hold at $\hat{x}$, then there exist finitely many indices $y^1, \ldots, y^q \in Y_0(\hat{x})$, non-negative scalars $\eta^\nu \in \mathbb{R}_+$ and $\tau_t^\nu \in \mathbb{R}_+$ for $\nu = 1, \ldots, q$ and $t \in T_0(\hat{x}, y^\nu)$, as well as positive scalars $\lambda_i \in \mathbb{R}_{++}$ for $i \in I$, satisfying $\sum_{i=1}^{p} \lambda_i = 1$ and (10).*

Proof. Since CDQ implies CCQ by (13), we can repeat the proof of Theorem 3.7 to find points $y^\nu \in Y_0(\hat{x})$ and pairs $(\alpha^\nu, \beta^\nu) \in F(\hat{x}, y^\nu)$ for $\nu = 1, \ldots, q$, non-negative scalars $\lambda_i \in \mathbb{R}_+$ for $i \in I$, and non-negative scalars $\gamma_\nu \in \mathbb{R}_+$ for $\nu = 1, \ldots, q$, such that $\sum_{i=1}^{p} \lambda_i = 1$ and

$$0_n \in \sum_{i=1}^{p} \lambda_i \partial f_i(\hat{x}) - \left(\sum_{\nu=1}^{q} \gamma_\nu \Big(\alpha^\nu \partial_x g(\hat{x}, y^\nu) + \sum_{t \in T_0(\hat{x}, y^\nu)} \beta_t^\nu \partial_x h_t(\hat{x}, y^\nu) \Big) \right).$$

Thus, there exist some $\hat{\xi}^i \in \partial f_i(\hat{x})$, for $i \in I$, and some

$$\hat{\zeta}^\nu \in \alpha^\nu \partial_x g(\hat{x}, y^\nu) + \sum_{t \in T_0(\hat{x}, y^\nu)} \beta_t^\nu \partial_x h_t(\hat{x}, y^\nu),$$

for $\nu = 1, \ldots, q$, such that

$$\sum_{i=1}^{p} \lambda_i \hat{\xi}^i - \sum_{\nu=1}^{q} \mu_\nu \hat{\zeta}^\nu = 0_n. \tag{14}$$

We claim that

$$\lambda_i > 0, \quad \text{for all} i \in I. \tag{15}$$

Otherwise, suppose $\lambda_k = 0$ for some $k \in I$. Since CDQ holds at $\hat{x}$, there exists $u \in \mathbb{R}^n$ such that

$$u \in \left(\bigcup_{i \in I \setminus k} \partial f_i(x_0) \right)^< \cap \big(\Omega(x_0)\big)^>.$$

This implies that

$$\begin{cases} \langle \hat{\xi}^\nu, u \rangle < 0, & \text{for all} \quad i \in I \setminus \{k\}, \\ \\ \langle \hat{\zeta}^\nu, u \rangle > 0, & \text{for all} \quad \nu = 1, \dots, q. \end{cases}$$

The above inequalities, together with (14), the condition $\sum_{i \in I \setminus \{k\}} \lambda_i = 1$, and the fact that $(\lambda_i, \mu_\nu) \in \mathbb{R}_+ \times \mathbb{R}_+$ for $i \in I$ and $\nu = 1, \dots, q$, imply that

$$0 = \underbrace{\lambda_k \langle \hat{\xi}^j, u \rangle}_{=0} + \underbrace{\sum_{i \in I \setminus \{k\}} \lambda_i \langle \hat{\xi}^i, u \rangle}_{<0} - \underbrace{\sum_{\nu=1}^{q} \mu_\nu \langle \hat{\zeta}^\nu, u \rangle}_{\geq 0} < 0.$$

This contradiction establishes that the claim (15) holds, thereby completing the proof. ■

It is worth noting that a strong KKT-type necessary condition for smooth MGSIPs at a *properly efficient solution* (which is stronger than weakly efficient solutions) in [23, Theorem 3.3]. However, neither Theorem 3.8 nor [23, Theorem 3.3] is strictly stronger than the other, even in the smooth case. In Theorem 3.8, the efficiency notion is weaker than in [23, Theorem 3.3], whereas the qualification condition in Theorem 3.8 is stronger. Consequently, these results can be regarded as two complementary, parallel theorems.

As the final point, we note that CDQ is strictly stronger than CCQ, and Theorem 3.8 may fail under the CCQ assumption. The following example illustrates this point.

Example 3.9. Consider problem (P) with the following data:

$$\begin{aligned} &T = \{1\}, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad f_1(x) = -x_1, \quad f_2(x) = x_2, \\ &g(x,y) = \min\{2 - y_1, 2 + y_1, 2 - y_2, 2 + y_2\}, \\ &h_1(x,y) = (y_1 - x_1)^2 + (y_2 - x_2)^2 - 1. \end{aligned}$$

It can be verified that $S = [-1,1] \times [-1,1]$ and $Y(x) = [-2,2] \times [-2,2]$ for all $x \in S$. A short calculation shows that $\hat{x} = (1,1)$ is a weakly efficient solution, and that $Y_0(\hat{x}) = \{\dot{y}, \ddot{y}\}$, with $\dot{y} = (1,2)$ and $\ddot{y} = (2,1)$. Since $T_0(\hat{x}, \dot{y}) = T_0(\hat{x}, \ddot{y}) = \{1\}$, the corresponding multiplier sets are

$$F(\hat{x}, \dot{y}) = \big\{ (\alpha, \beta) \in \mathbb{R}^2_+ \mid \alpha(0,-1) + \beta(0,2) = 0_2,\ \alpha + \beta = 1 \big\} = \left\{ \left(\frac{2}{3}, \frac{1}{3} \right) \right\},$$

$$F(\hat{x}, \ddot{y}) = \big\{ (\alpha, \beta) \in \mathbb{R}^2_+ \mid \alpha(-1,0) + \beta(2,0) = 0_2,\ \alpha + \beta = 1 \big\} = \left\{ \left(\frac{2}{3}, \frac{1}{3} \right) \right\}.$$

Furthermore, we can see

$$\Omega(\hat{x}) = \left\{\frac{2}{3}0_2 + \frac{1}{3}(0,-2)\right\} \cup \left\{\frac{2}{3}0_2 + \frac{1}{3}(-2,0)\right\} = \left\{(0,-\frac{2}{3}),(-\frac{2}{3},0)\right\},$$

$$\left(\bigcup_{i\in I\setminus\{1\}} \partial f_i(\hat{x})\right)^{<} \cap \left(\Omega(\hat{x})\right)^{>} = (-\infty,0)\times(-\infty,0),$$

$$\left(\bigcup_{i\in I\setminus\{2\}} \partial f_i(\hat{x})\right)^{<} \cap \left(\Omega(\hat{x})\right)^{>} = \emptyset.$$

On the other hand, $\Psi(x) = 0$ for all $x \in S$, and so, the PP condition holds at $\hat{x}$.

From these observations, we conclude that CDQ fails at $\hat{x}$, whereas GCCQ is satisfied. Furthermore, a direct computation shows that there do not exist positive scalars $\lambda_1, \lambda_2 \in \mathbb{R}_{++}$ and non-negative scalars $\mu_1, \mu_2 \in \mathbb{R}_+$ satisfying the strong KKT condition in Theorem 3.8:

$$\lambda_1\{(-1,0)\} + \lambda_2\{(0,1)\} - \mu_1\{(0,-\frac{2}{3})\} - \mu_2\{(-\frac{2}{3},0)\} = \{0_2\}. \tag{16}$$

However, Theorem 3.7 allows for non-strictly positive multipliers. Indeed, one can choose, for instance,

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad \mu_1 = \frac{3}{2}, \quad \mu_2 = 0,$$

which satisfies (16). This example thus demonstrates that Theorem 3.8 may fail under CCQ, while the weaker KKT condition in Theorem 3.7 remains valid.

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