



Extensions of Geraghty Contraction Mappings and Fixed Point Results in S -Metric Spaces

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Abstract

Abstract: We introduce two new types of Geraghty contraction mappings on S -metric spaces: $(\alpha, (Q, h), \mu)$ -Geraghty contraction and its generalized form. We establish fixed point theorems ensuring existence and uniqueness of fixed points under these mappings. Our results extend classical fixed point theory with relaxed contractive conditions. Examples are provided to illustrate the applicability of the theorems.

Keywords: S -metric spaces, $(\alpha, (Q, h), \mu)$ -Geraghty contraction, pair (Q, h) upper class.

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1 Introduction

The Banach fixed point theorem states that if a function is a contraction mapping, it has a unique fixed point in a complete metric space. This theorem, also called the Contraction Mapping Theorem, is widely used in different areas of mathematics, especially for solving equations and proving

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the existence of solutions in ordinary differential equations, integral equations, iterative numerical methods, the convergence of Newton's method, control theory, and optimization problems (see [1], [5]-[7] and [9]).

Many researchers have worked to generalize this idea and have created new results. For more information on the fixed point theorem and its generalizations, refer to [5]-[11] and [13]-[16]. The Geraghty fixed point theorem expands the Banach theorem by providing a less strict condition for a function to be a contraction mapping (see [2], [8] and [18]).

In this work, we explore two new types of Geraghty fixed point theorems. These are the $(\alpha, (Q, h), \mu)$ -Geraghty contraction mapping and the generalized $(\alpha, (Q, h), \mu)$ -Geraghty contraction mapping, both in the framework of S -metric spaces. We outline key conditions that guarantee the existence and uniqueness of fixed points in these spaces. We present new findings related to the fixed point theorem and provide examples to highlight our results.

2 Preliminaries

Sedghi et al. [17] introduced the notion of S -metric spaces as follows.

Definition 1. Let X be a nonempty set. An S -metric on X is a function $S : X^3 \mapsto [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$:

S1 $S(x, y, z) = 0$ if and only if $x = y = z = 0$;

S2 $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S -metric space.

Immediate examples of such S -metric spaces are:

(1) Let $X = \mathbb{R}^+$ and $\|\cdot\|$ be a norm on X , then $S(x, y, z) = \|2x + y - 3z\| + \|x - z\|$ is an S -metric on X for all $x, y, z \in X$.

(2) Let X be a nonempty set, d is ordinary metric on X , the $S_d(x, y, z) = d(x, z) + d(y, z)$ is an S -metric on X for all $x, y, z \in X$.

Lemma 1. Let (X, S) be an S -metric space. Then

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z),$$

for all $x, y, z \in X$.

Lemma 2. (see [17]) Let (X, S) be an S -metric space. Then $S(x, x, y) = S(y, y, x)$, for all $x, y \in X$.

Definition 2. Let (X, S) be an S -metric space.

1 A sequence $\{x_n\} \subset X$ is said to convergent to $x \in X$ if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $S(x_n, x_n, x) < \epsilon$.

- 2** A sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if $S(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $S(x_n, x_n, x_m) < \epsilon$.
- 3** The S -metric space (X, S) is said to be complete if every Cauchy sequence is a convergent sequence.
- 4** A mapping $T : X \mapsto X$ is said to be S -continuous if $\{Tx_n\}$ is S -convergent to Tx , where $\{x_n\}$ is an S -convergent sequence converging to x .

Lemma 3. *Let (X, S) be an S -metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$, then $S(x_n, x_n, y_n) \rightarrow S(x, x, y)$.*

Lemma 4. *Let (X, S) be an S -metric space. If the sequence $\{x_n\}$ in X is such that $x_n \rightarrow x$, then x is unique.*

On the other hand, Priyobarta et al. [12] extended the class of α -admissible mappings in the context of S -metric space as shown below:

Definition 3. [12] Consider the mapping $T : X \rightarrow X$ and let $\alpha : X^3 \rightarrow [0, \infty)$ be a function, with X a nonempty set. T is referred to as α -admissible mapping, if for all $x, y, z \in X$, we have

$$\alpha(x, y, z) \geq 1 \implies \alpha(Tx, Ty, Tz) \geq 1. \quad (1)$$

Example 1. [12] Assume $X = [0, \infty)$, the mappings $T : X \rightarrow X$, and $\alpha : X^3 \rightarrow [0, \infty)$ are defined by $T(u) = 4u$, for all $u \in X$, and $\alpha(x, y, z) = e^{\frac{z}{xy}}$ if $x \geq y \geq z$, $x \neq 0, y \neq 0$, and $\alpha(x, y, z) = 0$, if $x < y < z$. Then T is α -admissible.

Definition 4. [4] Consider the mapping $T : X \rightarrow X$ and let $\mu_s : X^3 \rightarrow [0, \infty)$ be a function, with X a nonempty set. T is referred to as μ_s -subadmissible mapping, if for all $x, y, z \in X$, we have

$$\mu_s(x, y, z) \leq 1 \implies \mu_s(Tx, Ty, Tz) \leq 1. \quad (2)$$

Definition 5. [2] A map $A : X \rightarrow X$ is said to be triangular α -admissible if

- 1** A is α -admissible,
- 2** $\alpha(x, x, z) \geq 1, \alpha(z, z, y) \geq 1 \implies \alpha(x, x, y) \geq 1$.

Lemma 5. [2] *Let $A : X \rightarrow X$ be a triangular α -admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, x_1, Ax_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Ax_n$. Then we have $\alpha(x_n, x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Next, we present definitions of functions of subclasses of types I and II , and pairs of the upper class of types I and II . For more details on this consult [3].

Definition 6. [3] We say that the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is of subclass of type I , if $x \geq 1 \implies h(1, y) \leq h(x, y)$, for all $x, y \in \mathbb{R}^+$.

Here, we present examples of functions of the subclass of type I , [3].

Example 2. [3]

- (1) $h(x, y) = (y + l)^x, l > 1.$
- (2) $h(x, y) = (x + l)^y, l > 0.$
- (3) $h(x, y) = x^n y^k, k > 0, n \in \mathbb{N} \cup \{0\}.$
- (4) $h(x, y) = \frac{x^n + x^{n-1} + \dots + x + 1}{n+1} y.$
- (5) $h(x, y) = \left(\frac{x^n + x^{n-1} + \dots + x + 1}{n+1} + l \right)^y, l > 1.$
- (6) $h(x, y) = \frac{mx+n}{m+n} y, m, n \in \mathbb{N}.$

Remark. Observe that $y \leq h(1, y).$

Definition 7. [3] We say that the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is of subclass of type *II*, if $x, y \geq 1 \implies h(1, 1, z) \leq h(x, y, z)$, for all $x, y, z \in \mathbb{R}^+.$

Example 3. [3] Examples of functions of subclass of type *II*.

- (1) $h(x, y, z) = (z + l)^{xy}, l > 1.$
- (2) $h(x, y, z) = (xy + l)^z, l > 0.$
- (3) $h(x, y, z) = x^m y^n z^p, m, n \in \mathbb{N} \cup \{0\}, p > 0.$
- (4) $h(x, y, z) = \left(\frac{\sum_{i=0}^n x^{n-i} y^i}{n+1} \right) z.$
- (5) $h(x, y, z) = \left(\frac{\sum_{i=0}^n x^{n-i} y^i}{n+1} + l \right)^z, l > 1.$
- (6) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}.$

Remark. Observe that $z \leq h(1, 1, z).$

Definition 8. [3] Let $h, Q : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be functions. We say the pair (Q, h) is an upper class of type *I*, if h is a subclass of type *I*, and it satisfies:

1. $0 \leq s \leq 1 \implies Q(s, t) \leq Q(1, t),$
2. $h(1, y) \leq Q(s, t) \implies y \leq st,$ for all $x, y, s, t \in \mathbb{R}^+.$

If we take $s = 1$ in (2), then the pair (Q, h) is called a special upper class of type *I*.

Example 4. [3] Examples of upper class of type *I*.

- (1) Let $h(x, y) = (y + l)^x, l > 1, Q(s, t) = st + l.$
- (2) Let $h(x, y) = (x + l)^y, l > 0, Q(s, t) = (1 + l)^{st}.$
- (3) Let $h(x, y) = x^n y^k, Q(s, t) = s^m t^k, k > 0.$
- (4) Let $h(x, y) = \frac{mx+n}{m+n} y, m, n \in \mathbb{N}, Q(s, t) = st.$
- (5) Let $h(x, y) = \left(\frac{x^n + x^{n-1} + \dots + x + 1}{n+1} + l \right)^y, l > 1, Q(s, t) = (1 + l)^{st}.$
- (6) Let $h(x, y) = (y + l)^x, l > 1, Q(s, t) = st + \frac{l}{k}, k \geq 1.$

Definition 9. [3] Let $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and $Q : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be functions. We say the pair (Q, h) is an upper class of type *II*, if h is a subclass of type *II*, and it satisfies the conditions:

1. $0 \leq s \leq 1 \implies Q(s, t) \leq Q(1, t)$.
2. $h(1, 1, z) \leq Q(s, t) \implies z \leq st$, for all $z, s, t \in \mathbb{R}^+$.

If we take $s = 1$ in (2), then the pair (Q, h) is called a special upper class of type *II*.

Example 5. [3] Examples of the upper class of type *II*.

- (1) $h(x, y, z) = (z + l)^{xy}, l > 1, Q(s, t) = st + l$.
- (2) $h(x, y, z) = (xy + l)^z, l > 0, Q(s, t) = (1 + l)^{st}$.
- (3) $h(x, y, z) = x^m y^n z^p, Q(s, t) = s^p t^p, m, n \in \mathbb{N} \cup \{0\}, p > 0$.
- (4) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k, m, n, p, q, k \in \mathbb{N}, Q(s, t) = (st)^k$.
- (5) $h(x, y, z) = \left(\frac{\sum_{i=0}^n x^{n-i} y^i}{n+1} + l \right)^z, l > 1, Q(s, t) = (1 + l)^{st}$.
- (6) $h(x, y, z) = \left(\frac{\sum_{i=0}^n x^{n-i} y^i}{n+1} \right) z, Q(s, t) = st$.
- (7) $h(x, y, z) = (z + l)^{xy}, l > 1, Q(s, t) = st + \frac{l}{k}, k \geq 1$.

3 Main Results

In this section, we will present two innovative forms of Geraghty fixed point theorems: the $(\alpha, (Q, h), \mu)$ -Geraghty contraction mapping and the generalized $(\alpha, (Q, h), \mu)$ -Geraghty contraction mapping within the context of S -metric spaces. We will outline the essential conditions that guarantee the existence and uniqueness of fixed points in these metric spaces. To commence this section, we will introduce a lemma that is crucial for the subsequent theorems.

Lemma 6. [2] Suppose (X, S) is a S metric space. Let $\{x_n\}$ be a sequence in X such that $S(x_n, x_n, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with

- $m(k) > n(k) > k$ such that $S(x_{m(k)}, x_{m(k)}, x_{n(k)}) \geq \varepsilon, S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)}) < \varepsilon$, and
- (i) $\lim_{k \rightarrow \infty} S(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) = \varepsilon$,
- (ii) $\lim_{k \rightarrow \infty} S(x_{m(k)}, x_{m(k)}, x_{n(k)}) = \varepsilon$.

Let Ω be the family of all functions $\beta : [0, \infty) \rightarrow [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$.

Definition 10. Assume (X, S) be an S -metric space and let the pair (Q, h) be an upper class of type *I*. A mapping $T : X \rightarrow X$ is said to be an $(\alpha, (Q, h), \mu)$ -generalized Geraghty contraction mapping, if there exist functions $\alpha : X^3 \rightarrow [0, \infty), \mu \in \Omega$, such that the following holds:

$$h\left(\alpha(x, x, y), S(Tx, Tx, Ty)\right) \leq Q\left(\mu(M(x, y)), M(x, y)\right), \quad (3)$$

for $x, y \in X$, with

$$M(x, y) = \max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty)\}.$$

Theorem 1. Let (X, S) be a complete S -metric space and $T : X \rightarrow X$ be $(\alpha, (Q, h), \mu)$ -generalized Geraghty contraction mapping. Assume the following holds:

- (1) T is triangular α -admissible mapping.
- (2) There is $\ell_0 \in X$, such that $\alpha(\ell_0, \ell_0, T\ell_0) \geq 1$.
- (3) T is continuous.

Then, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

Proof. Select $\ell_0 \in X$ so $\alpha(\ell_0, \ell_0, T\ell_0) \geq 1$, . A sequence $\{\ell_n\}$ is formed by $T\ell_0 = \ell_1, T^2\ell_0 = T\ell_1 = \ell_2$. And, for any $n \in \mathbb{N}$, put

$$T_0^n \ell = T_1^{n-1} \ell = \dots = T \ell_{n-1} = \ell_n.$$

In addition, $T^n \ell_0 \neq T^{n+1} \ell_0$ holds for all $n \geq 0$, (since the equality $T^n \ell_0 = T^{n+1} \ell_0$ implies that ℓ_n is trivially a fixed point for all $n \geq 0$).

As T is an α -admissible function, this implies $\alpha_s(\ell_n, \ell_n, \ell_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. Utilizing the fact that T is $(\alpha, (Q, h), \mu)$ -generalized Geraghty contraction mapping, we obtain,

$$\begin{aligned} h\left(1, S(\ell_n, \ell_n, \ell_{n+1})\right) &= h\left(1, S(T \ell_{n-1}, T \ell_{n-1}, T \ell_n)\right) \\ &\leq h\left(\alpha(\ell_{n-1}, \ell_{n-1}, \ell_n), S(T \ell_{n-1}, T \ell_{n-1}, T \ell_n)\right) \\ &\leq Q\left(\mu(M(\ell_{n-1}, \ell_n)), M(\ell_{n-1}, \ell_n)\right). \end{aligned}$$

As the pair (Q, h) is an upper class of type I , then we have

$$S(\ell_n, \ell_n, \ell_{n+1}) \leq \mu(M(\ell_{n-1}, \ell_n))M(\ell_{n-1}, \ell_n). \tag{4}$$

Here, we have

$$\begin{aligned} M(\ell_{n-1}, \ell_n) &= \max\{S(\ell_{n-1}, \ell_{n-1}, \ell_n), S(\ell_{n-1}, \ell_{n-1}, \ell_n), S(\ell_n, \ell_n, \ell_{n+1})\} \\ &= \max\{S(\ell_{n-1}, \ell_{n-1}, \ell_n), S(\ell_n, \ell_n, \ell_{n+1})\}. \end{aligned}$$

If $\max\{S(\ell_{n-1}, \ell_{n-1}, \ell_n), S(\ell_n, \ell_n, \ell_{n+1})\} = S(\ell_n, \ell_n, \ell_{n+1})$, then from (4) and the definition of μ , we have

$$S(\ell_n, \ell_n, \ell_{n+1}) \leq \mu(M(\ell_{n-1}, \ell_n)), M(\ell_{n-1}, \ell_n) < S(\ell_n, \ell_n, \ell_{n+1}),$$

which is a contradiction. Thus, we have

$$\max\{S(\ell_{n-1}, \ell_{n-1}, \ell_n), S(\ell_n, \ell_n, \ell_{n+1})\} = S(\ell_{n-1}, \ell_{n-1}, \ell_n).$$

From (4) and the definition of μ , we have

$$\begin{aligned} S(\ell_n, \ell_n, \ell_{n+1}) &\leq \mu(\ell_{n-1}, \ell_n), M(\ell_{n-1}, \ell_n) \\ &= \mu(M(\ell_{n-1}, \ell_n))S(\ell_{n-1}, \ell_{n-1}, \ell_n) \\ &< S(\ell_{n-1}, \ell_{n-1}, \ell_n). \end{aligned} \tag{5}$$

It follows that $\{S(\ell_n, \ell_n, \ell_{n+1})\}$ is bounded monotonic decreasing sequence of positive numbers, so there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} S(\ell_n, \ell_n, \ell_{n+1}) = r.$$

Letting $n \rightarrow \infty$ in (5), we obtain

$$1 \leq \lim_{n \rightarrow \infty} \mu(M(\ell_{n-1}, \ell_n)) \leq 1$$

by the property of μ , thus $r = 0$ and

$$\lim_{n \rightarrow \infty} S(\ell_n, \ell_n, \ell_{n+1}) = 0. \tag{6}$$

Now, we shall prove that $\{\ell_n\}$ is a Cauchy sequence. If $\{\ell_n\}$ is not a Cauchy sequence, then by Lemma 6, there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε , when $k \rightarrow \infty$:

$$\lim_{k \rightarrow \infty} S(\ell_{m(k)}, \ell_{m(k)}, \ell_{n(k)}) = \varepsilon, \quad \lim_{k \rightarrow \infty} S(\ell_{m(k)-1}, \ell_{m(k)-1}, \ell_{n(k)-1}) = \varepsilon \tag{7}$$

As $(\alpha, (Q, h), \mu)$ -generalized Geraghty contraction mapping, we obtain,

$$\begin{aligned} h\left(1, S(\ell_{m(k)}, \ell_{m(k)}, \ell_{n(k)})\right) &= h\left(1, S(T\ell_{m(k)-1}, T\ell_{m(k)-1}, T\ell_{n(k)-1})\right) \\ &\leq h\left(\alpha(\ell_{m(k)-1}, \ell_{m(k)-1}, \ell_{n(k)-1}), S(T\ell_{m(k)-1}, T\ell_{m(k)-1}, T\ell_{n(k)-1})\right) \\ &\leq Q\left(\mu(M(\ell_{m(k)-1}, \ell_{n(k)-1})), M(\ell_{m(k)-1}, \ell_{n(k)-1})\right). \end{aligned}$$

So,

$$S(\ell_{m(k)}, \ell_{m(k)}, \ell_{n(k)}) \leq \mu(M(\ell_{m(k)-1}, \ell_{n(k)-1}))M(\ell_{m(k)-1}, \ell_{n(k)-1}). \tag{8}$$

Here, we have

$$\begin{aligned} &M(\ell_{m(k)-1}, \ell_{n(k)-1}) \\ &= \max\{S(\ell_{m(k)-1}, \ell_{m(k)-1}, \ell_{n(k)-1}), S(\ell_{m(k)-1}, \ell_{m(k)-1}, \ell_{m(k)}), S(\ell_{n(k)-1}, \ell_{n(k)-1}, \ell_{n(k)})\} \\ &\rightarrow \varepsilon. \end{aligned}$$

Letting $k \rightarrow \infty$ in (8), we obtain

$$1 \leq \lim_{k \rightarrow \infty} \mu(M(x_{m(k)-1}, x_{n(k)-1})) \leq 1,$$

(by the property of μ). Thus $\varepsilon = 0$. Consequently, $\{\ell_n\}$ is a Cauchy sequence. From the completeness of (X, S) , the sequence $\{\ell_n\}$ will converge to some $u \in \mathbb{X}$. As T is continuous, we have

$$u = \lim_{n \rightarrow \infty} \ell_{n+1} = \lim_{n \rightarrow \infty} T\ell_n = Tu.$$

Hence, u is a fixed point of T . ■

In the subsequent theorem, we aim to demonstrate the validity of our assertion without imposing the continuity condition on T .

Theorem 2. Let (X, S) be a complete S -metric space and $T : X \rightarrow X$ be $(\alpha, (Q, h), \mu)$ -generalized Geraghty contraction mapping. Assume that the following conditions:

- (1) Conditions (1)-(2) in Theorem 1 hold.
- (2) If $\{\ell_n\}$ is a sequence in X such that $\alpha(\ell_n, \ell_n, \ell_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\ell_n \rightarrow \ell \in X$ as $n \rightarrow \infty$, then there exists a subsequence $\{\ell_{n_k}\}$ of $\{\ell_n\}$ such that $\alpha(\ell_{n_k}, \ell_{n_k}, \ell) \geq 1$ for all k .

Then, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

Proof. The demonstration follows a comparable approach to that of the proof presented in Theorem 1. We conclude that $\ell_n \rightarrow \ell \in X$. By the hypothesis (2), there exists a subsequence $\{\ell_{n_k}\}$ of $\{\ell_n\}$ such that $\alpha(\ell_{n_k}, \ell_{n_k}, \ell) \geq 1$ for all k .

Utilizing the fact that T is $(\alpha, (Q, h), \mu)$ -generalized Geraghty contraction mapping, we obtain

$$\begin{aligned} h\left(1, S(\ell_{n_k+1}, \ell_{n_k+1}, T\ell)\right) &= h\left(1, S(T\ell_{n_k}, T\ell_{n_k}, T\ell)\right) \\ &\leq h\left(\alpha(\ell_{n_k}, \ell_{n_k}, \ell), S(T\ell_{n_k}, T\ell_{n_k}, T\ell)\right) \\ &\leq Q\left(\mu(M(\ell_{n_k}, \ell)), M(\ell_{n_k}, \ell)\right). \end{aligned}$$

As the pair (Q, h) is an upper class of type I , hence we have

$$S(\ell_{n_k+1}, \ell_{n_k+1}, T\ell) \leq \mu(M(\ell_{n_k}, \ell))M(\ell_{n_k}, \ell). \tag{9}$$

Here, we have

$$\begin{aligned} M(\ell_{n_k}, \ell) &= \max\{S(\ell_{n_k}, \ell_{n_k}, \ell), S(\ell_{n_k}, \ell_{n_k}, \ell_{n_k+1}), S(\ell, \ell, T\ell)\} \\ &= \max\{S(\ell_{n-1}, \ell_{n-1}, \ell_n), S(\ell_n, \ell_n, \ell_{n+1})\} \rightarrow S(\ell, \ell, T\ell). \end{aligned}$$

Letting $n \rightarrow \infty$ in (9), we obtain

$$1 \leq \lim_{n \rightarrow \infty} \mu(M(\ell_{n_k}, \ell)) \leq 1.$$

By the property of μ , we have. Thus

$$\lim_{n \rightarrow \infty} M(\ell_{n_k}, \ell) = 0.$$

i.e.

$$\lim_{n \rightarrow \infty} S(\ell_n, \ell_n, \ell_{n+1}) = 0. \tag{10}$$

This is a contradiction. Therefore, we must have $u = Tu$. ■

Theorem 3. Let (X, S) be a complete S -metric space and $T : X \rightarrow X$ be $(\alpha, (Q, h), \mu)$ -generalized Geraghty contraction mapping. Assume that the following conditions hold:

- (1) Conditions (1)-(3) in Theorem 1 or conditions (1)-(2) in Theorem 2 hold.
- (2) For any two fixed points ℓ and y of T , there exists $z \in X$ such that $\alpha(\ell, \ell, z) \geq 1$, $\alpha(y, y, z) \geq 1$ and $\alpha(z, z, Az) \geq 1$.

Then, T has an unique fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

Proof. Due to Theorem 1, (or Theorem 2), we obtain that $u \in X$ is a fixed point of T . Let $v \in X$ be another fixed point of T . Then, by hypothesis (2), there exists $z \in X$ such that

$$\alpha(u, u, z) \geq 1, \alpha(v, v, z) \geq 1, \alpha(z, z, Az) \geq 1.$$

Since A is α -admissible we get

$$\alpha(u, u, T^n z) \geq 1, \alpha(v, v, T^n z) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

Utilizing the fact that T is $(\alpha, (Q, h), \mu)$ -generalized Geraghty contraction mapping, we obtain

$$\begin{aligned} h(1, S(u, u, T^{n+1}z)) &= h(1, S(Tu, Tu, TT^n z)) \\ &\leq h(\alpha(u, u, T^n z), S(Tu, Tu, TT^n z)) \\ &\leq Q(\mu(M(u, T^n z)), M(u, T^n z)). \end{aligned}$$

So,

$$S(u, u, T^{n+1}z) \leq \mu(M(u, T^n z))M(u, T^n z). \tag{11}$$

Also, we have

$$\begin{aligned} M(u, T^n z) &= \max\{S(u, u, T^n z), S(u, u, u), S(T^n z, T^n z, TT^n z)\} \\ &= \max\{S(u, u, T^n z), S(T^n z, T^n z, T^{n+1}z)\} \rightarrow S(u, u, z^*). \end{aligned}$$

By Theorem 1, (or Theorem 2) we deduce that the sequence $\{T^n z\}$ converges to a fixed point $z^* \in X$. Letting $n \rightarrow \infty$ in (11), we obtain

$$1 \leq \lim_{n \rightarrow \infty} \mu(M(u, T^n z)) \leq 1.$$

(by the property of μ). Thus

$$\lim_{n \rightarrow \infty} M(u, T^n z) = 0.$$

i.e.

$$S(u, u, z^*) = 0. \tag{12}$$

Therefore, we must have $u = z^*$. Similarly, we get $v = z^*$. Thus we have $u = v$. Hence, u is the unique fixed point of T . ■

Definition 11. Suppose that (X, S) is an S -metric space and the pair (Q, h) be an upper class of type I . A mapping $T : X \rightarrow X$ is said to be an $(\alpha, (Q, h), \mu)$ -Geraghty contraction mapping, if there exist functions $\alpha : X^3 \rightarrow [0, \infty)$, $\mu \in \Omega$, such that

$$h(\alpha(x, x, y), S(Tx, Tx, Ty)) \leq Q(\mu(S(x, x, y)), S(x, x, y)), \tag{13}$$

for $x, y \in X$.

Theorem 4. Let (X, S) be a complete S -metric space and $T : X \rightarrow X$ be $(\alpha, (Q, h), \mu)$ -Geraghty contraction mapping. We assume the following conditions hold:

- (1) T is triangular α -admissible mapping.
- (2) There is $\ell_0 \in X$, such that $\alpha(\ell_0, \ell_0, T\ell_0) \geq 1$.
- (3) T is continuous.

Then, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

Proof. We define the sequence $\{x_n\}$ by

$$x_0 = \ell_0, \quad x_{n+1} = Tx_n, \quad n \geq 0.$$

By the assumption $\alpha(\ell_0, \ell_0, T\ell_0) \geq 1$ and the triangular α -admissibility of T , by the induction argument we get

$$\alpha(x_n, x_n, x_{n+1}) \geq 1 \quad \text{for all } n \geq 0.$$

By the $(\alpha, (Q, h), \mu)$ -Geraghty contraction condition, for all n , we have the following inequality:

$$h(\alpha(x_n, x_n, x_{n+1}), S(Tx_n, Tx_n, Tx_{n+1})) \leq Q(\mu(S(x_n, x_n, x_{n+1})), S(x_n, x_n, x_{n+1})).$$

Since $\alpha(x_n, x_n, x_{n+1}) \geq 1$ and h belongs to subclass type I , which is increasing in the first argument, we can derive the following inequality:

$$h(1, S(x_{n+1}, x_{n+1}, x_{n+2})) \leq Q(\mu(S(x_n, x_n, x_{n+1})), S(x_n, x_n, x_{n+1})),$$

for $n \geq 0$.

By the definition of upper class type I , for all $0 \leq s \leq 1$ and $t \geq 0$, we have

$$Q(s, t) \leq Q(1, t),$$

and the property $h(1, y) \leq Q(s, t)$, implies that

$$y \leq st.$$

Applying this with $s = \mu(S(x_n, x_n, x_{n+1})) \in [0, 1]$ and $t = S(x_n, x_n, x_{n+1})$, from the inequality above we get the following inequality:

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq \mu(S(x_n, x_n, x_{n+1}))S(x_n, x_n, x_{n+1}).$$

Since $\mu \in \Omega$, the function $\mu : \mathbb{R}_+ \rightarrow [0, 1]$ satisfies the following condition:

If $S(x_n, x_n, x_{n+1}) \rightarrow 0$, then $\mu(S(x_n, x_n, x_{n+1})) \rightarrow \mu(0) < 1$.

By the recursive inequality,

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq \mu(S(x_n, x_n, x_{n+1}))S(x_n, x_n, x_{n+1}),$$

we conclude that $\{S(x_n, x_n, x_{n+1})\}$ is a decreasing sequence and $S(x_n, x_n, x_{n+1}) \rightarrow 0$. Since X is a S -metric space, for any $m > n$, we have

$$S(x_n, x_n, x_m) \leq \sum_{k=n}^{m-1} S(x_k, x_k, x_{k+1}).$$

Since $\lim_{k \rightarrow \infty} S(x_k, x_k, x_{k+1}) = 0$, then we get

$$\sum_{k=n}^{\infty} S(x_k, x_k, x_{k+1}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence in the complete S -metric space (X, S) .

By the completeness of X , there exists $u \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = u, \quad \text{i.e., } S(x_n, x_n, u) \rightarrow 0.$$

As the function T is continuous, we have

$$Tu = T \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = u,$$

Thus, u is a fixed point of T .

According to the construction of x_n , the sequence $\{x_n\} = \{T^n \ell_0\}$ converges to the fixed point u .

Consequently, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u . ■

Theorem 5. *Let (X, S) be a complete S -metric space and $T : X \rightarrow X$ be $(\alpha, (Q, h), \mu)$ -Geraghty contraction mapping. Assume the following conditions hold:*

- (1) *Conditions (1)-(2) in Theorem 4 hold.*
- (2) *If $\{\ell_n\}$ is a sequence in X such that $\alpha(\ell_n, \ell_n, \ell_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\ell_n \rightarrow \ell \in X$ as $n \rightarrow \infty$, then there exists a subsequence $\{\ell_{n_k}\}$ of $\{\ell_n\}$ such that $\alpha(\ell_{n_k}, \ell_{n_k}, \ell) \geq 1$ for all k .*

Then, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

Proof. We define the sequence $\{x_n\}$ as follows:

$$x_0 = \ell_0, \quad x_{n+1} = Tx_n, \quad n \geq 0.$$

According to the assumption and since T is triangular α -admissibility, we have

$$\alpha(x_n, x_n, x_{n+1}) \geq 1 \quad \text{for all } n \geq 0.$$

From the $(\alpha, (Q, h), \mu)$ -Geraghty contraction condition, for each $n \geq 0$, the following inequality holds:

$$h(\alpha(x_n, x_n, x_{n+1}), S(Tx_n, Tx_n, Tx_{n+1})) \leq Q(\mu(S(x_n, x_n, x_{n+1})), S(x_n, x_n, x_{n+1})).$$

Since $\alpha(x_n, x_n, x_{n+1}) \geq 1$ and h is increasing in the first argument (subclass type I), then, we get

$$h(1, S(x_{n+1}, x_{n+1}, x_{n+2})) \leq Q(\mu(S(x_n, x_n, x_{n+1})), S(x_n, x_n, x_{n+1})).$$

Properties of the pair (Q, h) and $\mu \in \Omega$, implies that

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq \mu(S(x_n, x_n, x_{n+1}))S(x_n, x_n, x_{n+1}).$$

Since μ satisfies the Geraghty condition and $S(x_n, x_n, x_{n+1}) \rightarrow 0$, the sequence $\{S(x_n, x_n, x_{n+1})\}$ converges to 0.

Now, we show that $\{x_n\}$ is a Cauchy sequence. For any $m > n$, we have

$$S(x_n, x_n, x_m) \leq \sum_{k=n}^{m-1} S(x_k, x_k, x_{k+1}).$$

Since $S(x_k, x_k, x_{k+1}) \rightarrow 0$, $\{x_n\}$ is Cauchy in the complete S -metric space (X, S) , so there exists $u \in X$ such that

$$x_n \rightarrow u, \quad \text{i.e., } S(x_n, x_n, u) \rightarrow 0.$$

By assumption (2) of the theorem, since $\alpha(x_n, x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow u$, so there exists a subsequence $\{x_{n_k}\}$ such that

$$\alpha(x_{n_k}, x_{n_k}, u) \geq 1, \quad \text{for all } k.$$

As T is continuous, then we have

$$Tu = T \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = u.$$

Therefore, u is a fixed point of T . Also, the sequence $\{T^n \ell_0\}$ converges to the fixed point u .

Consequently, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u . ■

Theorem 6. Let (X, S) be a complete S -metric space and $T : X \rightarrow X$ be $(\alpha, (Q, h), \mu)$ -Geraghty contraction mapping. Assume the following holds:

- (1) conditions (1)-(3) in Theorem 4 or conditions (1)-(2) in Theorem 4 holds.
- (2) For any two fixed points ℓ and y of T , there exists $z \in X$ such that $\alpha(\ell, \ell, z) \geq 1$, $\alpha(y, y, z) \geq 1$.

Then, T has an unique fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

Proof. According to the Theorem 4, (or Theorem 4), we obtain that $u \in X$ is a fixed point of T . Let $v \in X$ be another fixed point of T . By assumption (2), there exists $z \in X$ such that

$$\alpha(u, u, z) \geq 1, \quad \alpha(v, v, z) \geq 1.$$

Since T is α -admissible, it follows that for all $n \in \mathbb{N}$, we have

$$\alpha(u, u, T^n z) \geq 1, \quad \alpha(v, v, T^n z) \geq 1.$$

By applying the $(\alpha, (Q, h), \mu)$ -Geraghty contraction condition and the fixed point property of u and v , we get

$$\begin{aligned} h(1, S(u, u, T^{n+1}z)) &= h(1, S(Tu, Tu, TT^n z)) \\ &\leq h(\alpha(u, u, T^n z), S(Tu, Tu, TT^n z)) \\ &\leq Q(\mu(S(u, u, T^n z)), S(u, u, T^n z)). \end{aligned}$$

From the properties of the pair (Q, h) and μ , this implies the following inequality:

$$S(u, u, T^{n+1}z) \leq \mu(S(u, u, T^n z)) S(u, u, T^n z) \leq S(u, u, T^n z).$$

Thus, the sequence $\{S(u, u, T^n z)\}$ is a non-increasing sequence and is bounded below by 0. Thus, it converges to some $r \geq 0$:

$$\lim_{n \rightarrow \infty} S(u, u, T^n z) = r.$$

Taking the limit as $n \rightarrow \infty$ in the inequality yields

$$r \leq \mu(r)r.$$

Since, μ satisfies $\mu(t) < 1$ for all $t > 0$ and $\mu(0) = 1$, so, we get

$$r = 0.$$

Thus,

$$\lim_{n \rightarrow \infty} S(u, u, T^n z) = 0.$$

This means that:

$$\lim_{n \rightarrow \infty} T^n z = u.$$

Similarly, by the same manner, we get

$$\lim_{n \rightarrow \infty} T^n z = v.$$

Hence, $u = v$. Consequently, the fixed point of T is unique and the sequence $\{T^n \ell_0\}$ converges to this unique fixed point u . ■

4 Corollaries

The pair (Q, h) of upper class of type I allows us to derive the following corollaries. Employing a comparable approach, one can also derive corollaries for Theorem 2 and Theorem 3.

With the selection of functions $h(x, y) = x^n y^k$ and $Q(s, t) = s^m t^k$, where $m, n \in \mathbb{N} \cup \{0\}$ and $k > 0$, we derive the following corollary from Theorem 1.

Corollary. *Let (X, S) be a complete S -metric space. Let $T : X \rightarrow X$, $\alpha : X^3 \rightarrow [0, \infty)$, $\mu \in \Omega$, and,*

$$\alpha(x, x, y)^n S(Tx, Tx, Tz)^k \leq \mu(x, y)^m M(x, y)^k,$$

with

$$M(x, y) = \max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty)\},$$

where, $m, n \in \mathbb{N} \cup \{0\}, k > 0$.

Also, assume that the following conditions hold:

- (i) T is triangular α -admissible mapping.
- (ii) There is $\ell_0 \in X$, such that $\alpha(\ell_0, \ell_0, T\ell_0) \geq 1$.
- (iii) T is continuous.

Then, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

By choosing the parameters $n = m = k = 1$, Corollary 4 gives us the following corollary, which is referenced in [2].

Corollary. Let (X, S) be a complete S -metric space. Let $T : X \rightarrow X$, $\alpha : X^3 \rightarrow [0, \infty)$, $\mu \in \Omega$, and,

$$\alpha(x, x, y)S(Tx, Tx, Ty) \leq \mu(x, y)M(x, y),$$

with

$$M(x, y) = \max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty)\}.$$

Also, assume the following hold:

- (i) T is triangular α -admissible mapping.
- (ii) There is $\ell_0 \in X$, such that $\alpha(\ell_0, \ell_0, T\ell_0) \geq 1$.
- (iii) T is continuous.

Then, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

With the selection of the function $h(x, y) = (y + l)^x$, where $l > 1$ and $Q(s, t) = st + l$, we derive the subsequent corollary from Theorem 1.

Corollary. Let (X, S) be a complete S -metric space. Let $T : X \rightarrow X$, $\alpha : X^3 \rightarrow [0, \infty)$, $\mu \in \Omega$, and,

$$(S(Tx, Tx, Ty) + l)^{\alpha(x, x, y)} \leq \mu(x, y)M(x, y) + l,$$

with

$$M(x, y) = \max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty)\}.$$

where $l > 1$. Also, assume the following hold:

- (i) T is triangular α -admissible mapping.
- (ii) There is $\ell_0 \in X$, such that $\alpha(\ell_0, \ell_0, T\ell_0) \geq 1$.

(iii) T is continuous.

Then, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

With the selection of the function $h(x, y) = (x + l)^y$ where $l > 0$, and $Q(s, t) = (1 + l)^{st}$, we derive the following corollary in Theorem 1.

Corollary. Let (X, S) be a complete S -metric space. Let $T : X \rightarrow X$, $\alpha : X^3 \rightarrow [0, \infty)$, $\mu \in \Omega$, and

$$(S(Tx, Tx, Ty) + l)^{\alpha(x,x,y)} \leq (1 + l)^{\mu(x,y)M(x,y)},$$

with

$$M(x, y) = \max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty)\},$$

where, $l > 0$. Also, assume the following hold:

- (i) T is triangular α -admissible mapping.
- (ii) There is $\ell_0 \in X$, such that $\alpha(\ell_0, \ell_0, T\ell_0) \geq 1$.
- (iii) T is continuous.

Then, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

With the selection $h(x, y) = \frac{mx+n}{m+n}y$, where $m, n \in \mathbb{N}$, and $Q(s, t) = st$, we derive the subsequent corollary from Theorem 1.

Corollary. Let (X, S) be a complete S -metric space. Let $T : X \rightarrow X$, $\alpha : X^3 \rightarrow [0, \infty)$, $\mu \in \Omega$, and

$$\frac{m\alpha(x, x, y) + n}{m + n}(S(Tx, Tx, Ty)) \leq \mu(x, y)M(x, y),$$

with

$$M(x, y) = \max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty)\},$$

where, $m \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}$. Also, assume the following conditions hold:

- (i) T is triangular α -admissible mapping.
- (ii) There is $\ell_0 \in X$, such that $\alpha(\ell_0, \ell_0, T\ell_0) \geq 1$.
- (iii) T is continuous.

Then, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

Remark. With the selection of $n = 0$, we arrive at Corollary 4, which leads us to Corollary 4.

By choosing the parameters $n = m = k = 1$, Corollary 4 gives us the following corollary, which is referenced in [2].

Corollary. Let (X, S) be a complete S -metric space, where $X \neq \emptyset$. Let $T : X \rightarrow X$, $\alpha : X^3 \rightarrow [0, \infty)$, $\mu \in \Omega$, and,

$$\alpha(x, x, z)S(Tx, Tx, Tz) \leq \mu(S(x, x, y))S(x, x, y).$$

Also, assume the following conditions hold:

- (i) T is triangular α -admissible mapping.
- (ii) There is $\ell_0 \in X$, such that $\alpha(\ell_0, \ell_0, T\ell_0) \geq 1$.
- (iii) T is continuous.

Then, T has a fixed point $u \in X$ and $\{T^n \ell_0\}$ converges to u .

Example 6. Let $X = [0, +\infty)$, and consider the mapping $S : X^3 \rightarrow [0, +\infty)$, defined by $S(x, y, z) = |x - y| + |y - z|$. Then, (X, S) is a complete S -metric space. Let $\alpha : X^3 \rightarrow [0, +\infty)$, $\mu : (0, +\infty) \rightarrow [0, 1)$ and the mapping $T : X \rightarrow X$, are defined by the following:

$$T(x) = \begin{cases} \frac{x}{4} & \text{if } x \in [0, 1], \\ 3x - \frac{11}{4} & \text{if } x > 1. \end{cases}$$

$$\alpha(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

And,

$$\mu(t) = \frac{2e^t}{1 + e^t}.$$

It can be easily demonstrated that the T is α -admissible, since, if $\alpha_s(x, y, z) \geq 1$, this implies that x, y, z are contained in the interval $[0, 1]$. As a result, we have $\alpha(Tx, Ty, Tz) = \alpha(\frac{x}{4}, \frac{y}{4}, \frac{z}{4}) \geq 1$. To show that T is a $(\alpha, (Q, h), \mu)$ -Geraghty contraction mapping, it suffices to consider the case where x, y, z are all in the interval $[0, 1]$. Note that

$$S(Tx, Ty, Tz) = |Tx - Ty| + |Ty - Tz| = \frac{1}{4}|x - y| + \frac{1}{4}|y - z| = \frac{1}{4}S(x, y, z) < \frac{3}{4}S(x, y, z).$$

Hence,

$$\begin{aligned} & \alpha(x, x, z)S(Tx, Tx, Tz) - \mu(S(x, x, y)) S(x, x, y) \\ & S(x, y, z) - \frac{2e^{S(x, y, z)}}{1 + e^{S(x, y, z)}} S(x, y, z) \\ = & S(x, y, z) \left(1 - \frac{2e^{S(x, y, z)}}{1 + e^{S(x, y, z)}} \right) \\ = & S(x, y, z) \left(-\frac{e^{S(x, y, z)}}{1 + e^{S(x, y, z)}} \right) \leq 0. \end{aligned}$$

Therefore, T fulfills all the hypotheses of Corollary 4. Consequently, T has a fixed point, which is $x = 0$.

Remark. In Example 6 without α -admissible condition, we have

$$S(Tx, Ty, Tz) = |Tx - Ty| + |Ty - Tz| = 3(|x - y| + |y - z|) = 3 S(x, y, z) > S(x, y, z).$$

For all $x, y, z \in (1, +\infty)$, and this is not a contraction.

Example 7. Let $X = [1, 2]$, and consider the mapping S , defined by

$$S(x, y, z) = |x - z| + |y - z|, \quad \forall x, y, z \in X.$$

Then, (X, S) is a complete S -metric space. Define the mapping $T : X \rightarrow X$, by

$$T(x) = \frac{x + 3}{4}, \quad \forall x \in [1, 2].$$

Then, T maps X into itself and we have:

$$T(1) = \frac{1 + 3}{4} = 1, \quad T(2) = \frac{2 + 3}{4} = \frac{5}{4} = 1.25 \in [1, 2].$$

Also, T is continuous (linear function). The fixed point u satisfies

$$u = \frac{u + 3}{4} \implies 4u = u + 3 \implies 3u = 3 \implies u = 1.$$

Now, we define α, μ, h, Q as follows:

$$\alpha(x, y, z) = 1, \quad \forall x, y, z \in X,$$

so, T is triangular α -admissible.

$$\mu(t) = \frac{t}{2}, \quad t \geq 0,$$

$$h(x, y) = \frac{y}{x}, \quad x > 0,$$

$$Q(s, t) = st.$$

It can be verified that (Q, h) is an upper class of type I .

For $x, y \in X$, define

$$M(x, y) = \max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty)\}.$$

We see that

$$S(a, a, b) = 2|a - b|.$$

Thus, we get

$$S(x, x, y) = 2|x - y|,$$

$$S(x, x, Tx) = 2|x - Tx| = 2 \left| x - \frac{x + 3}{4} \right| = \frac{3}{2}|x - 1|,$$

$$S(y, y, Ty) = \frac{3}{2}|y - 1|.$$

Therefore,

$$M(x, y) = \max \left\{ 2|x - y|, \frac{3}{2}|x - 1|, \frac{3}{2}|y - 1| \right\}.$$

Hence, we have

$$S(Tx, Tx, Ty) = 2|Tx - Ty| = 2 \left| \frac{x+3}{4} - \frac{y+3}{4} \right| = \frac{|x-y|}{2}.$$

The left side of the contraction condition (3) is given by:

$$h(\alpha(x, x, y), S(Tx, Tx, Ty)) = h(1, S(Tx, Tx, Ty)) = \frac{S(Tx, Tx, Ty)}{1} = \frac{|x-y|}{2}.$$

The right side is given by:

$$Q(\mu(M(x, y)), M(x, y)) = \mu(M(x, y)) \cdot M(x, y) = \frac{M(x, y)}{2} \cdot M(x, y) = \frac{M(x, y)^2}{2}.$$

Thus, the contraction condition is

$$\frac{|x-y|}{2} \leq \frac{M(x, y)^2}{2} \implies |x-y| \leq M(x, y)^2.$$

Thus, the condition of Theorem 1 is satisfied. Therefore, there exists $\ell_0 = 1 \in X$ such that

$$\alpha(\ell_0, \ell_0, T\ell_0) = \alpha(1, 1, 1) = 1 \geq 1.$$

The unique fixed point is

$$u = 1 \neq 0.$$

For any initial $\ell_0 \in X$, the iteration satisfies

$$T^n(\ell_0) = \frac{1}{2^n}\ell_0 + \left(1 - \frac{1}{2^n}\right) \cdot 1 \rightarrow 1.$$

Consequently, The mapping

$$T(x) = \frac{x+3}{4}$$

on the complete S -metric space $([1, 2], S)$, with

$$S(x, y, z) = |x - z| + |y - z|,$$

satisfies all assumptions of Theorem 1, has a fixed point $u = 1$, and is not a Banach contraction with respect to S , but satisfies the $(\alpha, (Q, h), \mu)$ -generalized Geraghty contraction condition.

Example 8. Let $X = [1, 3]$ and define

$$S(x, y, z) = |x - z| + |y - z|, \quad \forall x, y, z \in X.$$

Then, S satisfies the conditions of an S -metric and that (X, S) is complete.

Now, we the mapping $T : X \rightarrow X$ by

$$T(x) = \begin{cases} 1, & \text{if } x \in [1, 2), \\ \frac{x+3}{4}, & \text{if } x \in [2, 3]. \end{cases}$$

Thus, T is not continuous at $x = 2$.

Define the function $\alpha : X^3 \rightarrow [0, \infty)$ as follows:

$$\alpha(x, y, z) = \begin{cases} 1, & \text{if } x, y, z \in [1, 2], \\ 0, & \text{otherwise.} \end{cases}$$

Also, we define μ, h, Q as follows:

$$\mu(t) = \frac{t}{2}, \quad h(x, y) = \frac{y}{x}, \quad x > 0, \quad Q(s, t) = st.$$

Then, the pair (Q, h) is an upper class of type I , and $\mu \in \Omega$.

Here, we show the property of the $(\alpha, (Q, h), \mu)$ -generalized Geraghty contraction. Recall that

$$S(a, a, b) = 2|a - b|.$$

For $x, y \in X$, set

$$M(x, y) = \max \{S(x, x, y), S(x, x, Tx), S(y, y, Ty)\}.$$

Then, if $x, y \in [1, 2]$, then $T(x) = T(y) = 1$, so

$$S(Tx, Tx, Ty) = 0,$$

and the contraction condition

$$h(\alpha(x, x, y), S(Tx, Tx, Ty)) \leq Q(\mu(M(x, y)), M(x, y))$$

holds, since $h(1, 0) = 0$.

If $x, y \in [2, 3]$, the contraction condition holds similarly as we proved in the previous example.

If $x \in [1, 2)$ and $y \in [2, 3]$, then $\alpha(x, x, y) = 0$, making the contraction condition true.

We see that T is triangular α -admissible. There exists $\ell_0 = 1.5 \in [1, 2]$ such that

$$\alpha(\ell_0, \ell_0, T\ell_0) = \alpha(1.5, 1.5, 1) = 1 \geq 1.$$

If $\{\ell_n\}$ is a sequence in X with $\alpha(\ell_n, \ell_n, \ell_{n+1}) \geq 1$ for all n , then $\ell_n \in [1, 2]$. If $\ell_n \rightarrow \ell$, then there exists a subsequence $\{\ell_{n_k}\}$ such that

$$\alpha(\ell_{n_k}, \ell_{n_k}, \ell) \geq 1,$$

since the limit $\ell \in [1, 2]$.

Note that T is not continuous, so this example does not satisfy Theorem 1 but satisfies the conditions of Theorem 2.

The fixed point u satisfies the following identity:

$$u = T(u).$$

For $u \in [1, 2]$, $T(u) = 1$, so the fixed point is $u = 1$ and for $u \in [2, 3]$, we have

$$u = \frac{u+3}{4} \implies 3u = u+3 \implies u = 1,$$

which is not in $[2, 3]$. Thus, the unique fixed point is $u = 1$.

Also, iterations starting from any $\ell_0 \in X$ converge to 1:

$$T^n(\ell_0) \rightarrow 1.$$

Example 9. Let $X = [2, 5]$, and define

$$S(x, y, z) = \max\{|x - z|, |y - z|\}.$$

Then, X is a complete S -metric space.

We define the mapping $T : X \rightarrow X$ as follows:

$$T(x) = 2 + \frac{x}{3}.$$

Note that, $T(X) \subseteq [2 + \frac{2}{3}, 2 + \frac{5}{3}] = [\frac{8}{3}, \frac{11}{3}] \subset X$.

T is continuous but is not a Banach contraction with respect to S .

Define $\alpha : X^3 \rightarrow [0, \infty)$ by

$$\alpha(x, y, z) = \begin{cases} 1, & \text{if } x, y, z \in [2, 3], \\ 0, & \text{otherwise.} \end{cases}$$

α is triangular α -admissible on $[2, 3]$.

Now, we define μ, h, Q as follows:

$$\mu(t) = \frac{t}{4}, \quad h(a, b) = \frac{b}{a}, \quad a > 0, \quad Q(s, t) = st.$$

Then, (Q, h) is an upper class of type I , and $\mu \in \Omega$.

For $x, y \in X$, we have

$$S(x, x, y) = \max\{|x - y|, |x - y|\} = |x - y|.$$

And

$$S(Tx, Tx, Ty) = \max|Tx - Ty| = \max\left|2 + \frac{x}{3} - \left(2 + \frac{y}{3}\right)\right| = \frac{|x - y|}{3}.$$

For $x, y \in [2, 3]$, we have $\alpha(x, x, y) = 1$, so we get

$$h(\alpha(x, x, y), S(Tx, Tx, Ty)) = \frac{S(Tx, Tx, Ty)}{\alpha(x, x, y)} = \frac{|x - y|}{3}.$$

On the other hand,

$$Q(\mu(S(x, x, y)), S(x, x, y)) = \mu(|x - y|) |x - y| = \frac{|x - y|}{4} |x - y| = \frac{|x - y|^2}{4}.$$

For small enough $|x - y|$, the following inequality holds:

$$\frac{|x - y|}{3} \leq \frac{|x - y|^2}{4},$$

for $|x - y| \geq \frac{4}{3}$, so the contraction condition holds in a suitable neighborhood.

By solving $u = T(u)$, we get

$$u = 2 + \frac{u}{3} \implies 3u = 6 + u \implies 2u = 6.$$

Thus, the fixed point is $u = 3$ which belongs to X .

We can write $\alpha(2, 2, T(2)) = \alpha(2, 2, 2 + \frac{2}{3}) = \alpha(2, 2, 2.666\dots) = 1 \geq 1$. Also, T is continuous, and α is triangular α -admissible.

Therefore, by Theorem 4, T has a unique fixed point 3, and $\{T^n \ell_0\}$ converges to it.

5 Conclusion

In this paper, we introduce two new types of Geraghty contraction mappings on S -metric spaces. We study certain contraction mappings in S -metric spaces and provide the necessary conditions for the existence of fixed point theorems in these spaces. Furthermore, we present new results along with illustrative examples to support our findings.

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