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Research Article

Generalized Contractions and Matkowski's Fixed Point Theorem in $\mathcal{F}$-Metric Spaces

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Abstract

Abstract: This research enhances Matkowski's fixed point theorem to incorporate generalized contractions within $\mathcal{F}$-metric spaces. We establish a Ćirić-type fixed point theorem for maps that satisfy a generalized contraction condition, demonstrating the existence and uniqueness of fixed points under more relaxed assumptions. We also explain how our conclusions can be used in dynamic programming to show that functional equations have solutions. Our results expand existing theorems and provide new perspectives on fixed point theory in $\mathcal{F}$-metric spaces, offering a flexible framework for relaxing the conventional triangle inequality.
Keywords: $\mathcal{F}$-metric space, Contraction-type mapping, Matkowski's fixed point theorem, Fixed point theory, Dynamic programming.
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1 Introduction

The Banach contraction principle, an essential component of fixed point theory, is extensively utilized in nonlinear analysis. Jleli and Samet [13] presented $\mathcal{F}$-metric spaces, which extend traditional metric spaces by modifying the triangle inequality via a non-decreasing function $\mathcal{F}$. This flexibility makes $\mathcal{F}$-metric spaces especially good for looking at fixed point theorems in unusual settings. This research expands Matkowski's fixed point theorem to demonstrate its use with Ćirić-type generalized contractions in $\mathcal{F}$-metric spaces. We also demonstrate its use in a dynamic programming context and verify the fulfillment of the requirements stipulated by our principal theorem.

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1.1 Preliminary Definitions and Results

We review the key definitions and results for $\mathcal{F}$-metric spaces.

Definition 1.1. A function $\Delta : X \times X \to \mathbb{R}^+$ is a **semi-metric** if:

1. $\Delta(x, y) \geq 0$, and $\Delta(x, y) = 0$ if and only if $x = y$,

2. $\Delta(x, y) = \Delta(y, x)$.

It is an $\mathcal{F}$**-metric** if there exists a non-decreasing function $\mathcal{F} : (0, +\infty) \to \mathbb{R}$ and $\tau > 0$ such that for all $x, y \in X$ and every sequence $\{u_i\}_{i=1}^n$ with $u_1 = x$, $u_n = y$,

$$\mathcal{F}(\Delta(x, y)) \leq \mathcal{F}\left(\sum_{i=1}^{n-1} \Delta(u_i, u_{i+1})\right) + \tau,$$

and

$$\lim_{n\to\infty} \mathcal{F}(t_n) = -\infty \text{ if and only if } \lim_{n\to\infty} t_n = 0. \tag{1}$$

The space (X, Δ) is **complete** if every Cauchy sequence converges. A subset $Y \subset X$ is **bounded** if there exists $M > 0$ such that $\Delta(x, y) \leq M$ for all $x, y \in Y$.

Definition 1.2 (Picard Sequence and Bounded Orbits)**.** Let (X, Δ) be an $\mathcal{F}$-metric space and $T : X \to X$. The *Picard sequence* of T at $x \in X$ is $\mathcal{O}(x) = \{T^n x\}_{n=0}^\infty$, where $T^{n+1}x = T(T^n x)$, $T^0 x = x$. The orbit $\mathcal{O}(x)$ is *bounded* if it is bounded per Definition 1.1.

Definition 1.3. The metric Δ is **separately continuous** if

$$\Delta(y_n, y) \to 0 \implies \Delta(x, y_n) \to \Delta(x, y),$$

and **jointly continuous** if

$$\Delta(x_n, x) \to 0 \text{ and } \Delta(y_n, y) \to 0 \implies \Delta(x_n, y_n) \to \Delta(x, y).$$

Definition 1.4. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing and satisfy:

(Θ_1) $\theta^n(t) \to 0$ as $n \to \infty$ for each $t > 0$.

(Θ_2) $\sum_{n=0}^\infty \theta^n(t) < +\infty$ for each $t > 0$.

The sets of such mappings are denoted Θ_1 and Θ_2, respectively, with $\Theta_2 \subset \Theta_1$.

If $\theta \in \Theta_1$, then $\theta(t) < t$ for $t > 0$ (see [23]). The class Θ_1 is known as Matkowski's functions [20].

2 Main Results

We extend Matkowski's fixed point theorem to Ćirić-type contractions in $\mathcal{F}$-metric spaces.

Theorem 2.1. *Let (X, Δ) be a complete $\mathcal{F}$-metric space, $\theta \in \Theta_1$, and $T : X \to X$ satisfy:*

$$\Delta(Tx, Ty) \leq \theta(\max\{\Delta(x, y), \Delta(x, Tx), \Delta(y, Ty)\}) \quad \text{for all } x, y \in X. \tag{2}$$

Assume:

1. ***Continuity**: Either T is continuous, or Δ is separately continuous.*
2. ***Bounded Orbits**: T has bounded orbits.*

Then:

1. *For any $x \in X$, $\{T^n x\}_{n=1}^{\infty}$ converges to $\xi \in X$.*
2. *ξ is the unique fixed point of T.*

Proof. Let $x \in X$, and suppose $T^{n+1}x \neq T^n x$. Consequently,

$$\Delta(T^{n+1}x, T^n x) \leq \theta^n(\Delta(Tx, x)) \to 0 \text{ as } n \to \infty.$$

Assume that $\{T^n x\}$ does not constitute a Cauchy sequence. There are sequences $\{n(k)\}$, $\{m(k)\}$, and a constant $L > 0$ such that $\Delta(T^{n(k)}x, T^{m(k)}x) \geq L$. For a sequence $s(k)$ that diverges to infinity,

$$\mathcal{F}(\Delta(T^{s(k)+p}x, T^{s(k)}x)) \leq \mathcal{F}\left(\sum_{i=1}^{p} \Delta(T^{s(k)+i-1}x, T^{s(k)+i}x)\right) + \tau.$$

Given that $\sum_{i=1}^{p} \Delta(T^{s(k)+i-1}x, T^{s(k)+i}x) \to 0$, it follows that

$$\lim_{k\to\infty} \mathcal{F}(\Delta(T^{s(k)+p}x, T^{s(k)}x)) = -\infty,$$

suggesting that

$$\lim_{k\to\infty} \Delta(T^{s(k)+p}x, T^{s(k)}x) = 0. \tag{3}$$

For an $\mathcal{F}$-metric,

$$\begin{aligned} \mathcal{F}(\Delta(T^{n(k)}x, T^{m(k)}x)) \leq \mathcal{F}(&\Delta(T^{n(k)}x, T^{n(k)+p}x) + \Delta(T^{n(k)+p}x, T^{m(k)+p}x) \\ &+ \Delta(T^{m(k)+p}x, T^{m(k)}x)) + \tau. \end{aligned} \tag{4}$$

If $\liminf_{k\to\infty} \Delta(T^{n(k)+p}x, T^{m(k)+p}x) = 0$, there exists $t(k)$ such that

$$\lim_{k\to\infty} \Delta(T^{n(t(k))+p}x, T^{m(t(k))+p}x) = 0.$$

Using (4) with $k = t(k)$, taking the limit superior yields a contradiction ($L = 0$). Thus,

$$\liminf_{k\to\infty} \Delta(T^{n(k)+p}x, T^{m(k)+p}x) = \delta_j > 0 \text{ for } j = 1, \ldots, p.$$

Let $\delta = \min\{\delta_1, \ldots, \delta_p\}$. Then

$$\Delta(T^{n(k)+j}x, T^{m(k)+j}x) \leq \theta(\max\{\Delta(T^{n(k)+j-1}x, T^{m(k)+j-1}x), \\ \Delta(T^{n(k)+j-1}x, T^{n(k)+j}x), \Delta(T^{m(k)+j-1}x, T^{m(k)+j}x)\}). \tag{5}$$

There exists k_0 such that

$$\Delta(T^{n(k)+j}x, T^{m(k)+j}x) > \frac{\delta}{2}, \quad j = 1, \ldots, p, \quad k > k_0, \tag{6}$$

and

$$\Delta(T^{n(k)+j-1}x, T^{n(k)+j}x), \Delta(T^{m(k)+j-1}x, T^{m(k)+j}x) < \frac{\delta}{2}, \quad k > k_0. \tag{7}$$

From (5), (6), and (7),

$$\Delta(T^{n(k)+j}x, T^{m(k)+j}x) \leq \theta(\Delta(T^{n(k)+j-1}x, T^{m(k)+j-1}x)).$$

Thus,

$$\Delta(T^{n(k)+p}x, T^{m(k)+p}x) < \theta^p(\Delta(T^{n(k)}x, T^{m(k)}x)). \tag{8}$$

With $\Delta(T^{n(k)}x, T^{m(k)}x) \leq M$, we get

$$\Delta(T^{n(k)+p}x, T^{m(k)+p}x) \leq \theta^p(M). \tag{9}$$

Combining (4) and (9),

$$\mathcal{F}(\Delta(T^{n(k)}x, T^{m(k)}x)) \leq \mathcal{F}(\Delta(T^{n(k)}x, T^{n(k)+p}x) + \theta^p(M) + \Delta(T^{m(k)+p}x, T^{m(k)}x)) + \tau.$$

For $\epsilon > 0$, as $\theta^p(M) \to 0$,

$$L \leq \mathcal{F}(3\epsilon) + \tau.$$

Taking $\epsilon \to 0$, $L = 0$, a contradiction. Thus, $\{T^n x\}$ is Cauchy and converges to $z \in X$. If Δ or T is continuous,

$$\Delta(T^{n+1}x, Tz) \leq \theta(\max\{\Delta(T^n x, z), \Delta(T^n x, T^{n+1}x), \Delta(z, Tz)\}) \to \theta(\Delta(z, Tz)),$$

implying $z = Tz$. Uniqueness follows as in the original proof. ■

Corollary 2.2. *Let (X, Δ) be a complete $\mathcal{F}$-metric space, $\theta \in \Theta_1$, and $T : X \to X$ satisfy*

$$\Delta(Tx, Ty) \leq \theta(\Delta(x, y)).$$

If T has bounded orbits, then T has a unique fixed point z, and $\{T^n x\}$ converges to z.

Proof. Since T is continuous, the result follows from Theorem 2.1. ■

Remark 2.3. In Corollary 2.2, if T is non-expansive and has a fixed point ξ, then $\mathcal{O}(x)$ is bounded, and $\{T^n x\}$ converges to ξ.

3 Applications

Dynamic programming, introduced by Bellman [3], provides a framework for solving optimization problems involving sequential decisions, such as resource allocation, optimal control, and economic planning. A common formulation is the Bellman functional equation, which models the value function for multistage decision processes:

$$\mu(\omega) = \sup_{\varsigma \in D} \left\{ f(\omega, \varsigma) + K(\omega, \varsigma, \mu(g(\omega, \varsigma))) \right\}, \quad \omega \in \Phi_1,$$

where Φ_1 and Φ_2 are Banach spaces representing state and decision spaces, respectively; $f : \Phi_1 \times \Phi_2 \to \mathbb{R}$ is the immediate reward; $K : \Phi_1 \times \Phi_2 \times \mathbb{R} \to \mathbb{R}$ represents the future value; $g : \Phi_1 \times \Phi_2 \to \Phi_1$ is the state transition function; and $D \subseteq \Phi_2$ is the decision set. This equation, often solved using fixed point theorems [8], is central to applications in operations research and economics.

We apply Theorem 2.1 to find the unique bounded solution μ. Let $B(\Phi_1)$ be the space of bounded real-valued functions on Φ_1, equipped with the $\mathcal{F}$-metric

$$\Delta(h, k) = \sup_{\iota \in \Phi_1} |h(\iota) - k(\iota)|, \quad \mathcal{F}(t) = \ln t, \quad \tau = 1.$$

Assume f and K are bounded, and

$$|K(\omega, \varsigma, h(\omega)) - K(\omega, \varsigma, k(\omega))| \leq e^{-a(|h(\omega) - k(\omega)|)}, \quad a > 1.$$

Define the operator

$$V(h)(\omega) = \sup_{\varsigma \in \Phi_2} \left\{ f(\omega, \varsigma) + K(\omega, \varsigma, h(g(\omega, \varsigma))) \right\}.$$

By Theorem 2.1, with $\theta(t) = e^{-at} \in \Theta_1$, V satisfies the contraction condition

$$\Delta(Vh, Vk) \leq \theta(\Delta(h, k)) = e^{-a\Delta(h,k)},$$

ensuring a unique fixed point in $B(\Phi_1)$, which is the unique bounded solution to the functional equation. This demonstrates the applicability of $\mathcal{F}$-metric spaces in solving practical optimization problems [11].

Proposition 3.1. *Let $(B(\Phi_1), \Delta)$ be the space of bounded real-valued functions on a Banach space Φ_1, with the $\mathcal{F}$-metric defined by $\Delta(h, k) = \sup_{\iota \in \Phi_1} |h(\iota) - k(\iota)|$, where $\mathcal{F}(t) = \ln t$ and $\tau = 1$. Let $V : B(\Phi_1) \to B(\Phi_1)$ be an operator defined by $V(h)(\omega) = \sup_{\varsigma \in \Phi_2} \{f(\omega, \varsigma) + K(\omega, \varsigma, h(g(\omega, \varsigma)))\}$, where f and K are bounded functions, and $|K(\omega, \varsigma, t_1) - K(\omega, \varsigma, t_2)| \leq e^{-a|t_1 - t_2|}$ for $a > 1$. Then, V satisfies the conditions of Theorem 2.1, as:*

1. *$(B(\Phi_1), \Delta)$ is a complete $\mathcal{F}$-metric space.*
2. *V is a contraction, satisfying $\Delta(Vh, Vk) \leq e^{-a\Delta(h,k)} = \theta(\Delta(h, k))$, where $\theta(t) = e^{-at}$.*
3. *$\theta(t) = e^{-at} \in \Theta_1$.*
4. *Both Δ and V are continuous.*
5. *The orbits of V are bounded.*

Consequently, V has a unique fixed point in $B(\Phi_1)$, which is the unique bounded solution to the functional equation $\mu(\omega) = \sup_{\varsigma\in\Phi_2} \{f(\omega,\varsigma) + K(\omega,\varsigma,\mu(g(\omega,\varsigma)))\}$.

Proof. We verify each condition of Theorem 2.1.

(1) $\mathcal{F}$-Metric Space and Completeness. The space $(B(\Phi_1),\Delta)$ is defined with $\Delta(h,k) = \sup_{\iota\in\Phi_1} |h(\iota) - k(\iota)|$, $\mathcal{F}(t) = \ln t$, and $\tau = 1$. For any $h, k \in B(\Phi_1)$ and sequence $\{u_i\}_{i=1}^n \subset B(\Phi_1)$ with $u_1 = h$, $u_n = k$,

$$\mathcal{F}(\Delta(h,k)) = \ln\left(\sup_{\iota\in\Phi_1} |h(\iota) - k(\iota)|\right).$$

Since $|h(\iota) - k(\iota)| \le \sum_{i=1}^{n-1} |u_i(\iota) - u_{i+1}(\iota)|$, taking the supremum,

$$\sup_{\iota\in\Phi_1} |h(\iota) - k(\iota)| \le \sum_{i=1}^{n-1} \sup_{\iota\in\Phi_1} |u_i(\iota) - u_{i+1}(\iota)| = \sum_{i=1}^{n-1} \Delta(u_i, u_{i+1}).$$

As $\mathcal{F}(t) = \ln t$ is non-decreasing, $\mathcal{F}(\Delta(h,k)) \le \mathcal{F}\left(\sum_{i=1}^{n-1} \Delta(u_i,u_{i+1})\right) + \tau$. Moreover, $\lim_{t\to 0^+} \mathcal{F}(t) = \lim_{t\to 0^+} \ln t = -\infty$, so Δ is an $\mathcal{F}$-metric. Since $B(\Phi_1)$ is a Banach space under the sup-norm, $(B(\Phi_1),\Delta)$ is complete.

(2) Contraction Condition. We show $\Delta(Vh, Vk) \le \theta(\Delta(h,k))$, where $\theta(t) = e^{-at}$, $a > 1$. For $h, k \in B(\Phi_1)$,

$$\Delta(Vh, Vk) = \sup_{\omega\in\Phi_1} |V(h)(\omega) - V(k)(\omega)|,$$

where

$$V(h)(\omega) - V(k)(\omega) = \sup_{\varsigma\in\Phi_2} \{f(\omega,\varsigma) + K(\omega,\varsigma,h(g(\omega,\varsigma)))\} - \sup_{\varsigma\in\Phi_2} \{f(\omega,\varsigma) + K(\omega,\varsigma,k(g(\omega,\varsigma)))\}.$$

For any $\epsilon > 0$, there exists $\varsigma' \in \Phi_2$ such that

$$V(k)(\omega) - \epsilon \le f(\omega,\varsigma') + K(\omega,\varsigma',k(g(\omega,\varsigma'))) \le V(k)(\omega).$$

Thus,

$$V(h)(\omega) - V(k)(\omega) \le K(\omega,\varsigma',h(g(\omega,\varsigma'))) - K(\omega,\varsigma',k(g(\omega,\varsigma'))) + \epsilon.$$

Given $|K(\omega,\varsigma',t_1) - K(\omega,\varsigma',t_2)| \le e^{-a|t_1-t_2|}$, and since $|h(g(\omega,\varsigma')) - k(g(\omega,\varsigma'))| \le \Delta(h,k)$, we have

$$|V(h)(\omega) - V(k)(\omega)| \le e^{-a\Delta(h,k)} + \epsilon.$$

Taking the supremum over $\omega \in \Phi_1$ and letting $\epsilon \to 0$, $\Delta(Vh, Vk) \le e^{-a\Delta(h,k)} = \theta(\Delta(h,k))$, satisfying the contraction condition.

(3) $\theta \in \Theta_1$. For $\theta(t) = e^{-at}$, compute iterates: $\theta^0(t) = t$, $\theta^1(t) = e^{-at}$. Assuming $\theta^{n-1}(t) = e^{-a^{n-1}t}$, we get

$$\theta^n(t) = e^{-a\cdot e^{-a^{n-1}t}} = e^{-a^n t}.$$

Since $a > 1$, $a^n \to \infty$, so $\theta^n(t) \to 0$ for $t > 0$, hence $\theta \in \Theta_1$.

(4) Continuity. For Δ, if $k_n \to k$, i.e., $\Delta(k_n, k) \to 0$, then $\Delta(h, k_n) \to \Delta(h,k)$ due to uniform convergence in the sup-norm. For V, if $h_n \to h$, then

$$\Delta(Vh_n, Vh) \le e^{-a\Delta(h_n,h)} \to 0,$$

so V is continuous.

(5) Bounded Orbits. Since $|f(\omega,\varsigma)| \le M_1$, $|K(\omega,\varsigma,t)| \le M_2$,

$$|V(h)(\omega)| \le \sup_{\varsigma} |f(\omega,\varsigma) + K(\omega,\varsigma,h(g(\omega,\varsigma)))| \le M_1 + M_2.$$

Thus, the orbit $\{V^n h\}_{n=0}^{\infty}$ is bounded, as $\Delta(V^n h, V^m h) \le 2(M_1 + M_2)$.

Since $(B(\Phi_1), \Delta)$ is a complete $\mathcal{F}$-metric space, V is a continuous contraction with $\theta \in \Theta_1$, and has bounded orbits, Theorem 2.1 implies V has a unique fixed point, the unique bounded solution to $\mu(\omega) = \sup_{\varsigma\in\Phi_2} \{f(\omega,\varsigma) + K(\omega,\varsigma,\mu(g(\omega,\varsigma)))\}$. ∎

3.1 Applications to Nonlinear Integral Equations

Fixed point theorems in $\mathcal{F}$-metric spaces provide powerful tools for establishing the existence and uniqueness of solutions to nonlinear integral equations, particularly those of the Volterra or Fredholm type. Here, we apply Corollary 2.2 to prove the existence of a unique solution to a nonlinear Volterra-type integral equation of the second kind.

Consider the integral equation

$$\mu(t) = f(t) + \int_a^t K(t,s,\mu(s))\,ds, \quad t \in [a,b], \tag{10}$$

where $f : [a,b] \to \mathbb{R}$ is a continuous function, and $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous kernel satisfying a generalized contraction condition.

Let $X = C([a,b],\mathbb{R})$, the space of continuous real-valued functions on $[a,b]$, equipped with the $\mathcal{F}$-metric

$$\Delta(h,k) = \sup_{t\in[a,b]} |h(t) - k(t)|, \quad \mathcal{F}(t) = \ln t, \quad \tau = 1.$$

As shown in Section 3, (X,Δ) is a complete $\mathcal{F}$-metric space.

Define the operator $T : X \to X$ by

$$T(h)(t) = f(t) + \int_a^t K(t,s,h(s))\,ds.$$

Assume that the kernel K satisfies the following condition:

$$|K(t,s,u) - K(t,s,v)| \le \alpha e^{-a|u-v|}, \quad \forall t,s \in [a,b],\ u,v \in \mathbb{R}, \tag{11}$$

where $a > 1$ and $\alpha > 0$ is a constant such that $\alpha(b-a) < 1$. Additionally, assume f and K are bounded to ensure T maps X into itself.

For $h,k \in X$, compute the distance between $T(h)$ and $T(k)$:

$$\Delta(Th,Tk) = \sup_{t\in[a,b]} \left| \int_a^t [K(t,s,h(s)) - K(t,s,k(s))]\,ds \right|.$$

Using (11),

$$\left| \int_a^t [K(t,s,h(s)) - K(t,s,k(s))]\,ds \right| \le \int_a^t |K(t,s,h(s)) - K(t,s,k(s))|\,ds \le \alpha \int_a^t e^{-a|h(s)-k(s)|}\,ds.$$

Since $|h(s) - k(s)| \leq \Delta(h, k)$, and e^{-ar} is decreasing, we have

$$e^{-a|h(s)-k(s)|} \leq e^{-a\Delta(h,k)}.$$

Thus,

$$\Delta(Th, Tk) \leq \alpha \int_a^t e^{-a\Delta(h,k)}\, ds \leq \alpha(t-a)e^{-a\Delta(h,k)} \leq \alpha(b-a)e^{-a\Delta(h,k)}.$$

Define $\theta(r) = \alpha(b-a)e^{-ar}$. Since $\alpha(b-a) < 1$, and for $\theta^n(r) = \alpha(b-a)e^{-a\theta^{n-1}(r)}$, we have $\theta^n(r) \to 0$ as $n \to \infty$ (as $a > 1$), so $\theta \in \Theta_1$. Thus,

$$\Delta(Th, Tk) \leq \theta(\Delta(h, k)).$$

The operator T is continuous due to the continuity of f and K, and the orbits of T are bounded since f and K are bounded:

$$|T(h)(t)| \leq |f(t)| + \int_a^t |K(t, s, h(s))|\, ds \leq M_f + (b-a)M_K,$$

where $M_f = \sup_{t\in[a,b]} |f(t)|$, $M_K = \sup_{t,s\in[a,b],u\in\mathbb{R}} |K(t, s, u)|$. Hence, $\Delta(T^n h, T^m h) \leq 2(M_f + (b-a)M_K)$.

By Corollary 2.2, T has a unique fixed point $\mu \in X$, which is the unique continuous solution to (10).

This result extends prior work on integral equations in generalized metric spaces [24, 11, 1], showcasing the robustness of $\mathcal{F}$-metric spaces in handling nonlinear kernels with exponential decay.

4 Conclusion

We extended Matkowski's fixed point theorem to Ćirić-type contractions in $\mathcal{F}$-metric spaces, with Theorem 2.1 and Corollary 2.2 generalizing existing results. The application to dynamic programming demonstrates the practical utility of our findings, with a verification section reserved for confirming the operator's properties.

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