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Research Article

A New Approach to Existence of Best Proximity Points for Mixed Multivalued Maps in Partial Metric Spaces

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Abstract

Abstract: The goal of this short note is to show that the main result of M. Aslantas [M. Aslantas, Finding a solution to an optimization problem and an application, J. Optim. Theory Appl., 194, 121–141 (2022)] which is related to existence of best proximity points for multivalued non-self mappings in the setting of partial metric spaces is a particular conclusion of a fixed point theorem due to J. Ahmed et al. [J. Ahmad, A, Azam and M, Arshad, Fixed points of multivalued mappings in partial metric spaces, Fixed Point Theory Appl., 2013:316].
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1 Introduction and Preliminaries

Let Υ be a nonempty set and $\sigma : \Upsilon \times \Upsilon \to [0, \infty)$ be a function. Then the function σ is called a *partial metric* on Υ provided that

(i) $\sigma(u, u) = \sigma(v, v) = \sigma(u, v) \Leftrightarrow u = v$,

(ii) $\sigma(u, u) \le \sigma(u, v)$,

(iii) $\sigma(u, v) = \sigma(v, u)$,

(iv) $\sigma(u, v) \le \sigma(u, w) + \sigma(w, v) - \sigma(w, w)$,

for any $u, v, w \in \Upsilon$. In this case, we say that the pair (Υ, σ) is a partial metric space.

Let (Υ, σ) be a partial metric space and define $d_\sigma : \Upsilon \times \Upsilon \to [0, \infty)$ with

$$d_\sigma(u, v) := 2\sigma(u, v) - \sigma(u, u) - \sigma(v, v),$$

for any $u, v \in \Upsilon$. Then (Υ, d_σ) is a metric space.

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Also, each partial metric σ on Υ generates a T_0 topology τ_σ on Υ with a base of the family of open σ-balls

$$\Big\{\mathcal{N}_\sigma(u,r) : u \in \Upsilon, r > 0\Big\},$$

where $\mathcal{N}_\sigma(u,r) := \{v \in \Upsilon : \sigma(u,v) < r + \sigma(u,u)\}$.

Let (Υ, σ) be a partial metric space and $\{u_n\}$ be a sequence in Υ. Then the sequence $\{u_n\}$ is said to be convergent to an element $p \in \Upsilon$ whenever $\lim_{n\to\infty} \sigma(u_n, u) \to \sigma(u,u)$. Moreover, the sequence $\{u_n\}$ is called a Cauchy sequence provided that $\lim_{m,n\to\infty} \sigma(u_n, u_m)$ exists and is finite. We say that the partial metric space (Υ, σ) is *complete* if every Cauchy sequence in Υ converges to an element of Υ.

A sequence $\{u_n\}$ is called 0-Cauchy sequence if $\lim_{m,n\to\infty} \sigma(u_m, u_n) = 0$. Also, a partial metric space (υ, σ) is called 0-complete partial metric space if every 0-Cauchy sequence converges to a point $u \in \Upsilon$ w.r.t. τ_σ such that

$$\lim_{m,n\to\infty} \sigma(u_m, u_n) = \sigma(u,u) = 0.$$

Clearly, every complete partial metric space is a 0-complete partial metric space, but the reverse may not be hold, necessarily.

Throughout this paper the set of all nonempty, bounded and closed subset of a partial metric space (Υ, σ) is denoted by $CB^\sigma(\Upsilon)$. The partial Hausdorff metric on $CB^\sigma(\Upsilon)$ is defined with

$$H_\sigma(U,V) := \max\Big\{\sup_{u\in U} \sigma(u,V), \sup_{v\in V} \sigma(v,U)\Big\}, \quad \forall\, U, V \in CB^\sigma(\Upsilon),$$

where $\sigma(u,V) = \inf\{\sigma(u,v) : v \in V\}$. We refer to [1] for more information about the properties of Hausdorff metric spaces.

Notation. A class of all functions $\mu : [0,\infty) \to [0,1)$ which satisfy the condition $\limsup_{t\to s^+} \mu(t) < 1$ for all $s \geq 0$ will be denoted by Φ.

The following fixed point theorem was proved in [2].

Theorem 1.1. (see Theorem 13 of [2]) *Let (Υ, σ) be a complete partial metric space and $S : \Upsilon \to CB^\sigma(\Upsilon)$ be a multivalued mapping such that*

$$H_\sigma(Sx_1, Sx_2) \leq \mu\big(\sigma(x_1,x_2)\big)\sigma(x_1,x_2), \qquad \forall x_1, x_2 \in \Upsilon,$$

where $\mu \in \Phi$. Then S has a fixed point.

Remark 1.2. It is worth noticing that in [3] Romaguera gave an example showing that, in a partial metric space (Υ, σ), the set $CB^\sigma(\Upsilon)$ may be empty. To address this problem, he introduced the notion of a *mixed multivalued mapping* $S : \Upsilon \to \Upsilon \cup CB^\sigma(\Upsilon)$ on a partial metric space (Υ, σ). According to this new approach either $S(x)$ is a singleton, i.e. $|S(x)| = 1$, or $S(x) \in CB^\sigma(\Upsilon)$ for all $x \in \Upsilon$. In this case, for any subset A of υ, the image of A under the mixed multivalued mapping S is defined as

$$S(A) := \bigcup_{x\in A} S(x).$$

The non-self version of mixed multivalued contractions was considered in [4] as below.

Definition 1.3. Let (Υ, σ) be a partial metric space and U, V be nonempty subsets of Υ. A mapping $T : U \to V \cup CB^{\sigma}(V)$ is said to be a mixed multivalued non-self MT-contraction provided that there exists $\mu \in \Phi$ such that

$$H_{\sigma}(Tu_1, Tu_2) \leq \mu\big(\sigma(u_1, u_2)\big)\sigma(u_1, u_2), \quad \forall u_1, u_2 \in U.$$

Now assume the U and V are two nonempty subsets of a partial metric space (Υ, σ) such that $U \cap V = \emptyset$ and $T : U \to CB^{\sigma}(V)$ is a multivalued non-self mapping. In this case the fixed point problem $u \in Tu$ does not have a solution. In this situation, a point $u^{\star} \in U$ is said to be a *best proximity point* of T if

$$\sigma(u^{\star}, Tu^{\star}) = \sigma(U, V) := \inf\big\{\sigma(u, v) : (u, v) \in u \times V\big\}.$$

Just recently in [4], Theorem 1.1 was generalized to non-self mappings in order to study the existence of best proximity points. Before stating the main conclusion of [4] we recall the following notions.

For a nonempty pair (U, V) of subsets of a partial metric space (Υ, σ) we set

$$U_0 := \big\{u \in U : \sigma(u, v) = \sigma(U, V), \quad \text{for some } v \in V\big\},$$

$$V_0 := \big\{v \in V : \sigma(u, v) = \sigma(U, V), \quad \text{for some } u \in U\big\}.$$

Definition 1.4. Let (Υ, σ) be a partial metric space and U, V be two nonempty subsets of Υ such that $U_0 \neq \emptyset$. The pair (U, V) is said to have weak P^*-property whenever

$$\begin{cases} \sigma(u_1, v_1) = \sigma(U, V), \\ \sigma(u_2, v_2) = \sigma(U, V), \end{cases} \Rightarrow \sigma(u_1, u_2) \leq \sigma(v_1, v_2),$$

for all $u_1 \neq u_2 \in U_0$ and $v_1, v_2 \in V_0$.

Here, we state the main result of [4].

Theorem 1.5. (see Theorem 3.1 of [4]) *Let (Υ, σ) be a 0-complete partial metric space and U, V be two nonempty closed subsets of Υ w.r.t. τ_{σ} such that $U_0 \neq \emptyset$ and (U, V) has the weak P^*-property. Assume that $T : U \to V \cup CB^{\sigma}(V)$ is a mixed multivalued non-self* MT*-contraction satisfying $T(U_0) \subseteq V_0$. Then T has a best proximity point $u^{\star} \in U$.*

The main purpose of this article is to show that Theorem 1.5 is a particular case of Theorem 1.1 and so the results of [4] cannot be considered as extensions of fixed point theory.

2 Main Results

Theorem 2.1. *Theorem 1.5 is a straightforward consequence of Theorem 1.1.*

Proof. Note that if $U \cap V \neq \emptyset$ Then by Theorem 1.1, T has a fixed point in $U \cap V$ and we are finished. So assume that $U \cap V = \emptyset$. Define the function $d : \Upsilon \times \Upsilon \to [0, \infty)$ with

$$d(u, v) = \begin{cases} \sigma(u, v), & \text{if} \quad u \neq v; \\ 0, & \text{if} \quad u = v. \end{cases}$$

From Propositions 4.8.8 and 4.8.10 of [5] (Υ, σ) is 0-complete if and only if (Υ, d) is a complete metric space and $\tau_{d_\sigma} \subseteq \tau_d$. Since $U \cap V = \emptyset$, for any $(u, v) \in U \times V$ we have

$$\sigma(u, v) = d(u, v), \quad \sigma(u, V) = d(u, V) := \inf\{d(u, v) : v \in V\},$$
$$\sigma(U, V) = d(U, V) := \inf\{d(u, v) : (u, v) \in U \times V\}.$$

We now organize the proof by the following steps:
♠ *The set U_0 is nonempty and closed w.r.t. metric d.*
Proof. It is sufficient to note that

$$U_0 = \bigcap_{n=1}^{\infty} \Big\{u \in U : d(u, V) \leq d(U, V) + \frac{1}{n}\Big\}.$$

In this situation, U_0 is complete w.r.t. d and so is 0-complete.
♠ *A multivalued mapping $S : U_0 \to U_0 \cup CB^d(U_0)$ defined by*

$$S(x) := \big\{u \in U_0 : \ \exists\ v \in T(x);\ d(u, T(x)) = d(u, v) = d(U, V)\big\}, \quad \forall x \in U_0,$$

is a mixed multivalued MT*-contraction.*
Proof. At first assume that $x \in U_0$. Since $T(U_0) \subseteq V_0$, we have $Tx \subseteq V_0$. By the fact that $Tx \neq \emptyset$, we can consider an element $v \in Tx$. Thus there exists a point $u \in U_0$ for which $d(u, v) = d(U, V)$ and so,

$$d(u, Tx) \leq d(u, v) = d(U, V) \leq d(u, Tx),$$

which ensures that $u \in S(x)$ and hence $S(x) \neq \emptyset$. Let $\varepsilon > 0$ be given and $x, y \in U_0$. By the definition of S, for $u \in S(x)$ there is a point $w \in T(x)$ such that $d(u, w) = d(U, V)$. Since H_σ is a partial Hausdorff metric based on the partial metric σ, there is an element $z \in T(y)$ such that

$$\sigma(w, z) \leq H_\sigma\big(T(x), T(y)\big) + \varepsilon, \tag{1}$$

Again, by definition of the mapping S, there is a point $v \in S(y)$ such that $d(v, z) = d(U, V)$. In view of the fact that (U, V) has the weak P^*-property,

$$\sigma(u, v) \leq \sigma(w, z). \tag{2}$$

It now follows from (1) and (2) that for each $x, y \in U_0$ and $u \in S(x)$ there exists $v \in S(y)$ for which

$$\sigma(u, v) \leq H_\sigma\big(T(x), T(y)\big) + \varepsilon. \tag{3}$$

Since $\varepsilon > 0$ is arbitrary and by the definition of H_σ, we conclude

$$H_\sigma\big(S(x), S(y)\big) \leq H_\sigma\big(T(x), T(y)\big).$$

Because T satisfies

$$H_\sigma\big(T(x), T(y)\big) \leq \mu\big(\sigma(x, y)\big)\sigma(x, y),$$

by (3) we obtain

$$H_\sigma\big(S(x), S(y)\big) \leq \mu\big(\sigma(x, y)\big)\sigma(x, y),$$

where $\mu \in \Phi$, that is, S is a mixed multivalued MT-contraction.

Since S satisfies all of the conditions of Theorem 1.1, it has a fixed point $q \in U_0$, that is, $q \in S(q)$. Again using the definition of S we obtain

$$\sigma\big(q, T(q)\big) = d\big(q, T(q)\big) = d(U, V) = \sigma(U, V),$$

and so q is a best proximity point of T.

■

3 Conclusion

We employed a new approach to the existence of best proximity points for a class of multivalued non-self contractions and discussed on the results of [4] by considering the corresponding fixed point theorems in [2].

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