Zagros J. Appl. Math. & Data. Anal.

Available online at `https://zjamda.abru.ac.ir/`
ISSN 0000-0000
Vol. 1, No. 1, 2025, 1–16
Research Article

Construction of Reproducing Kernel Functions Using Chebyshev Polynomials

Mohsen Khaleghi
Department of Mathematics, Fa.C., Islamic Azad University, Fasa

Saeid Abbasbandy*
Department of Applied Mathematics, Faculty of Science,
Imam Khomeini International University, Qazvin

Esmail Babolian
Department of Computer Science, Faculty of Mathematical Sciences and Computer,
Kharazmi University, Tehran

Abstract

Abstract: This paper presents a polynomial-based reproducing kernel method (RKM) for approximating solutions to two-point boundary value problems. The approach constructs a kernel function using shifted Chebyshev polynomials, yielding solutions in the form of a truncated series. A key contribution is mitigating the dependence of convergence on dense nodal discretization—a common limitation in traditional RKMs. Numerical results demonstrate that the proposed method achieves comparable or superior accuracy with fewer nodes, significantly lowering computational costs.

Keywords: Hilbert Space, Reproducing kernel method, Chebyshev polynomials.
2020 Mathematics Subject Classification: 65L10; 46E22; 35C10

1 Introduction

The concept of reproducing kernel methods (RKMs) was first introduced by S. Zaremba in the early 20th century, originally applied to boundary value problems involving harmonic and biharmonic functions [1]. Since then, RKMs have been employed in a wide range of mathematical and computational contexts, including ordinary differential equations [14, 16, 17, 23, 24, 25], boundary value problems [36], and partial differential equations [7, 8, 9, 12, 26, 27, 28, 29, 30, 31, 32, 33, 34, 37],

---

*Corresponding author: abbasbandy@yahoo.com; abbasbandy@ikiu.ac.ir

integral and integro-differential equations [10, 12, 33, 34], and various classes of nonlinear and singular boundary value problems [2, 4, 5, 6, 15, 16, 17, 18, 19, 20]. Beyond traditional applications, they have also found use in areas such as nonlinear analysis [3] and machine learning [13]. Despite their broad utility, classical RKMs often rely heavily on increasing the number of nodal points to ensure convergence, which can lead to numerical instability and high computational cost due to repeated orthogonalization steps. In this work, we aim to mitigate this limitation by introducing a novel approach that constructs a reproducing kernel based on shifted Chebyshev polynomials. This construction not only adheres to the necessary boundary conditions but also yields a more stable and efficient solution framework. The structure of the present paper is organized as follows. In the following section, we review the reproducing kernel methods. In Section 3, we will introduce a proper closed form of orthonormal basis functions that independently satisfy the homogeneous boundary conditions on $[a, b]$ and a polynomial reproducing kernel function. Section 4 introduces our method as a Chebyshev reproducing kernel method (C-RKM). We show that C-RKM does not need a dense sequence of nodal points. Examples, including singular problem, singularly perturbed problem and singularly perturbed problem with an interior layer are given to illustrate the applicability and accuracy in Section 5. We end the paper with a few conclusions in Section 6.

2 A glance at reproducing kernel method

The implementation of the reproducing kernel method (RKM) for solving differential equations typically involves four main steps:

Step 1, Selecting an appropriate solution space is crucial, as an ill-suited space may hinder the ability to find an accurate solution.

Step 2, Constructing the reproducing kernel function, which is usually obtained by solving a boundary value problem, followed by solving a corresponding linear system.

Step 3, Generating an orthonormal basis for the solution space. This involves using the reproducing kernel, applying the boundary operator, choosing a sufficiently dense sequence of nodal points, and executing a Gram-Schmidt orthogonalization procedure.

Step 4, Representing the exact solution as an infinite sum of the orthonormal basis functions obtained in the previous step. For practical computations, this series is truncated after N terms to yield an approximate solution.

3 Basis functions and polynomial reproducing kernel function

3.1 Basis functions

The shifted Chebyshev polynomials of the first kind, defined on a general interval $[a, b]$, can be generated using the following recurrence relations:

$$T_0(x) = 1, \quad T_1(x) = \frac{2x - (a+b)}{b-a},$$

$$T_n(x) = 2(\frac{2x - (a+b)}{b-a})T_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots.$$

The orthogonality condition is

$$\langle T_n, T_m \rangle = \int_a^b w_{[a,b]}(x) T_n(x) T_m(x)\, dx = \begin{cases} 0, & n \neq m, \\ \frac{(b-a)\pi}{2}, & n = m = 0, \\ \frac{(b-a)\pi}{4}, & n = m \neq 0, \end{cases} \tag{1}$$

where

$$w_{[a,b]}(x) = \frac{1}{\sqrt{1 - (\frac{2x-a-b}{b-a})^2}}. \tag{2}$$

In solving the boundary value problems, the practical use of those basis functions can be useful that, independently, satisfy the boundary conditions. So we have to construct Chebyshev basis functions that independently satisfy the homogeneous boundary conditions conditions such as

$$u(a) = u(b) = 0. \tag{3}$$

Lemma 3.1. *[21] The functions defined by*

$$\varphi_n(x) = \begin{cases} T_n(x) - T_0(x), & \text{for even } n, \\ T_n(x) - T_1(x), & \text{for odd } n, \end{cases} \quad n \geq 2, \tag{4}$$

have the property

$$\varphi_n(a) = \varphi_n(b) = 0, \quad \text{for all } n,$$

and for the functions space satisfying the boundary conditions (3), the basis functions defined by (4) are complete.

Proposition 3.2. *Applying Gram-Schmidt process to* $\{\varphi_n\}_{n=2}^{\infty}$*, by inner product (1), we obtain an orthonormal basis* $\{h_n\}_{n=2}^{\infty}$ *of functions on* $[a, b]$ *that satisfy the boundary conditions (3). It can be shown that* h_i*, for* $i = 2, 3, \ldots$*, has the following closed form*

$$h_i(x) = 2\sqrt{\frac{(i-1)}{(i+1)(b-a)\pi}} \begin{cases} T_i(x) - \frac{2}{i-1}\sum_{k=1}^{\frac{i-2}{2}} T_{2k}(x) - \frac{1}{i-1}, & \text{for even } i, \\ T_i(x) - \frac{2}{i-1}\sum_{k=1}^{\frac{i-1}{2}} T_{2k-1}(x), & \text{for odd } i. \end{cases} \tag{5}$$

Proof. Use Lemma 3.1 and induction on i.

3.2 Polynomial reproducing kernel function

Definition 3.3. [2] For a nonempty set $\mathcal{X}$, let $(\mathcal{H}, \langle ., . \rangle_{\mathcal{H}})$ be a Hilbert space of real-valued functions on some set $\mathcal{X}$. A function $R : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ is said to be the *reproducing kernel function* of $\mathcal{H}$ if and only if

1. $R(x, .) \in \mathcal{H}$, $\forall x \in \mathcal{X}$,
2. $\langle \varphi(.), R(x, .) \rangle_{\mathcal{H}} = \varphi(x)$, $\forall \varphi \in \mathcal{H}$, $\forall x \in \mathcal{X}$ (reproducing property).

Also, a Hilbert space of functions $(\mathcal{H}, \langle ., . \rangle_{\mathcal{H}})$ that possesses a reproducing kernel R is a reproducing kernel Hilbert space (RKHS); we denote it by $(\mathcal{H}, \langle ., . \rangle_{\mathcal{H}}, R)$. In the following, we often denote by R_y the function $R(y, .) : t \longmapsto R(y, t)$.

Theorem 3.4. *[35] Every finite-dimensional inner product space is complete. Let M be a finite-dimensional subspace of the inner product space V then:*

1. *Each bounded sequence in M has a subsequence that converges to a point in M;*
2. *M is closed;*
3. *M is complete;*
4. *Suppose $\{e_1, e_2, \dots, e_n\}$ is a basis for M, $g_k = \sum_{i=1}^{n} \alpha_{ki} y_i$, and $g = \sum_{1}^{n} \alpha_i e_i$. Then $g_k \to g$ if and only if $\alpha_{ki} \to \alpha_i$ for $i = 1, 2, \dots, n$.*

Theorem 3.5. *[2] If $\mathcal{H}$ is an n-dimensional Hilbert space, $\{e_i\}_{i=1}^{n}$ is the orthonormal basis when $x \in X$ and X is an abstract set, then for any fixed $y \in X$,*

$$R_y(x) = \sum_{i=1}^{n} e_i(x)\bar{e}_i(y), \tag{6}$$

is the reproducing kernel function of $\mathcal{H}$.

Let $\Pi_w^m[a, b]$ denotes the weighted inner product space of polynomials on $[a, b]$ with real coefficients and degree less than or equal to m with inner product,

$$\langle u, v\rangle_{\Pi_w^m} = \int_a^b w_{[a,b]}(x)u(x)v(x)dx, \quad \forall u, v \in \Pi_w^m[a, b],$$

where $w_{[a,b]}(x)$ is defined by Eq. (2), and norm

$$||u||_{\Pi_w^m} = \sqrt{\langle u, u\rangle_{\Pi_w^m}}, \quad \forall u \in \Pi_w^m[a, b].$$

Theorem 3.6. *The function space $\Pi_w^m[a, b]$, by its inner product and norm (mentioned above), is a reproducing kernel Hilbert space.*

Proof. It is clear that $\Pi_w^m[a, b]$ is an $(m + 1)$-dimensional inner product space, so by Theorems 3.4 and 3.5, $\Pi_w^m[a, b]$ is a RKHS.

Theorem 3.7. *[2] Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of approximate solution to $u \in \Pi_w^m[a, b]$. If $\{u_n\}_{n=1}^{\infty}$ converges to u in the sense of $||.||_{\Pi_w^m}$, then $\{u_n\}_{n=1}^{\infty}$ converges to u uniformly.*

For the practical use of the RKM method, it is necessary to define a closed subspace of $\Pi_w^m[a, b]$ by imposing required homogeneous boundary conditions on it.

Definition 3.8. Let

$${}^{o}\Pi_w^m[a, b] = \{u \mid u \in \Pi_w^m[a, b], u(a) = u(b) = 0\},$$

so similar to the proof of Theorem 3.6 by using Eq.(4), we can prove that the function space ${}^{o}\Pi_w^m[a, b]$ is a reproducing kernel Hilbert space.

According to the Theorem 3.5 and Proposition 3.2, The polynomial reproducing kernel function $R_y^m(x)$ of ${}^o\Pi_w^m[a,b]$ has the expression:

$$R_y^m(x) = \sum_{i=2}^{m} h_i(x)h_i(y). \tag{7}$$

Eq. (7) shows that the polynomial reproducing kernel function $R_y^m(x)$ not only can easily be constructed by a finite sum of basis functions, also reproducing kernel Hilbert space $\Pi_w^m[a,b]$ can be updated by increasing m.

Theorem 3.9. *[2] The reproducing kernel space ${}^o\Pi_w^m[a,b]$ is a closed subspace of $\Pi_w^m[a,b]$.*

4 The Chebyshev-reproducing kernel method (C-RKM)

4.1 Representation of exact solution in ${}^o\Pi_w^m[a,b]$

Here, we develop a polynomial reproducing kernel method for solving the following two point boundary value problem in ${}^o\Pi_w^m[a,b]$

$$p(x)u'' + q(x)u' + r(x)u = f(x), \quad a \le x \le b,$$

subject to boundary conditions

$$u(a) = \alpha, \quad u(b) = \beta,$$

where $p(x), q(x), r(x) \in C(a,b)$ and $f(x) \in L_w^2[a,b]$ are sufficiently regular given functions and α, β are finite constants. We assume that the solution of the problem exists and is unique. Also, let the problem, after homogenization can be transformed into the following form:

$$\begin{cases} (\mathbb{L}u)(x) = f(x), & a \le x \le b, \\ u(a) = u(b) = 0, \end{cases} \tag{8}$$

where

$$\mathbb{L} :{}^o \Pi_w^m[a,b] \longrightarrow L_w^2[a,b],$$

is a bounded linear operator. We shall give the representation of analytical solution of Eq. (8) in the space ${}^o\Pi_w^m[a,b]$. Let $R_y^m(x)$ be the polynomial reproducing kernel function of ${}^o\Pi_w^m[a,b]$. For any fixed $x_i \in [a,b]$, put

$$\psi_i^m(x) = \mathbb{L}_y R_y^m(x)|_{y=x_i}, \tag{9}$$

where the subscript y in the operator $\mathbb{L}$ indicates that the operator $\mathbb{L}$ applies to the function y.

Theorem 4.1. *Let $\{x_i\}_{i=0}^{m-2}$ be any $(m-1)$-distinct points in the interval (a,b), then $\{\psi_i^m\}_{i=0}^{m-2}$ is a basis for ${}^o\Pi_w^m[a,b]$.*

Proof. For each fixed $u \in {}^o\Pi_w^m[a,b]$, let

$$\langle u(.), \psi_i^m(.)\rangle_{{}^o\Pi_w^m} = 0, \quad i = 0, 1, \dots, m-2,$$

which means that for $i = 0, 1, 2, \dots, m-2$,

$$0 = \langle u(.), \mathbb{L}_y R_y^m(.)|_{y=x_i}\rangle_{{}^o\Pi_w^m} = \mathbb{L}_y \langle u(.), R_y^m(.)\rangle_{{}^o\Pi_w^m}|_{y=x_i} = \mathbb{L}u(x_i).$$

So from the existence of $\mathbb{L}^{-1}$,

$$u(x_i)=0, \quad i=0,1,2,\ldots,m-2.$$

Since $u(a)=u(b)=0$ then $u\equiv 0$. Therefore, $\{\psi_i^m\}_{i=0}^{m-2}$ is a complete system for ${}^o\Pi_w^m[a,b]$.

Theorem 4.1 shows that, in our method (C-RKM), use of a finite sequence of nodal points is sufficient. So, implementation of C-RKM for solving problems does not need a dense sequence of nodal points.

The orthonormal system $\{\bar{\psi}_i^m\}_{i=0}^{m-2}$ of ${}^o\Pi_w^m[a,b]$ can be deduced from Gram-Schmidt orthogonalization process using $\{\psi_i^m\}_{i=0}^{m-2}$,

$$\bar{\psi}_i^m(x)=\sum_{k=0}^{i}\beta_{ik}^m\psi_k^m(x),$$

where β_{ik}^m are orthogonalization coefficients.

Theorem 4.2. *Suppose that u_m is the unique exact solution of Eq. (8) in ${}^o\Pi_w^m[a,b]$. Let $\{x_i\}_{i=0}^{m-2}$ be any $(m-1)$-distinct points in (a,b), then*

$$u_m(x)=\sum_{i=0}^{m-2}\sum_{k=0}^{i}\beta_{ik}^m f(x_k)\bar{\psi}_i^m(x). \tag{10}$$

Proof. Since $u_m\in{}^o\Pi_w^m[a,b]$ then by Theorem 4.1 we have

$$u_m(x)=\sum_{i=0}^{m-2}\langle u_m(.),\bar{\psi}_i^m(.)\rangle_{{}^o\Pi_w^m}\bar{\psi}_i^m(x)=\sum_{i=0}^{m-2}\langle u_m(.),\sum_{k=0}^{i}\beta_{ik}^m\psi_k^m(.)\rangle_{{}^o\Pi_w^m}\bar{\psi}_i^m(x)$$

$$=\sum_{i=0}^{m-2}\sum_{k=0}^{i}\beta_{ik}^m\langle u_m(.),\psi_k^m(.)\rangle_{{}^o\Pi_w^m}\bar{\psi}_i^m(x)=\sum_{i=0}^{m-2}\sum_{k=0}^{i}\beta_{ik}^m\langle u_m(.),\mathbb{L}_yR_y^m(.)\rangle_{{}^o\Pi_w^m}|_{y=x_k}\bar{\psi}_i^m(x)$$

$$=\sum_{i=0}^{m-2}\sum_{k=0}^{i}\beta_{ik}^m\mathbb{L}_y\langle u_m(.),R_y^m(.)\rangle_{{}^o\Pi_w^m}|_{y=x_k}\bar{\psi}_i^m(x)=\sum_{i=0}^{m-2}\sum_{k=0}^{i}\beta_{ik}^m\mathbb{L}_yu_m(y)|_{y=x_k}\bar{\psi}_i^m(x)$$

$$=\sum_{i=0}^{m-2}\sum_{k=0}^{i}\beta_{ik}^m f(x_k)\bar{\psi}_i^m(x).$$

4.2 Convergence analysis in ${}^oL_w^2[a,b]$

Let in Eq. (8), $\mathbb{L}:{}^o L_w^2[a,b]\longrightarrow L_w^2[a,b]$, be a bounded linear operator where

$${}^oL_w^2[a,b]=\{u \mid u\in L_w^2[a,b], u(a)=u(b)=0\}.$$

We assume that for any integer $m>1$, $\{x_i\}_{i=0}^{m-2}$ be any $(m-1)$-distinct points in (a,b). Let $u(x)\in{}^oL_w^2[a,b]$ and $u_m(x)\in{}^o\Pi_w^m[a,b]$ be the exact and approximate solutions of the problem. We discuss the convergence of the approximate solutions constructed in 10.

Theorem 4.3. *If $u \in {}^{o}L^2_w[a,b]$ and $u_m \in {}^{o}\Pi^m_w[a,b]$ is the approximate of u then*

$$||u_m - u||_{{}^{o}L^2_w} \longrightarrow 0, \quad m \longrightarrow \infty.$$

Moreover the sequence $||u_m - u||_{{}^{o}L^2_w}$ is monotonically decreasing in m.

Proof. From Lemma 3.1, Preposition 3.2 and Eq. (9), it follows that

$$u(x) = \sum_{i=2}^{\infty} \langle u, h_i \rangle_{{}^{o}L^2_w} h_i(x),$$

and for any integer m,

$$\langle h_j, \psi_i^m \rangle_{{}^{o}L^2_w} = 0, \quad i = 0, 1, \dots, m-2, \quad j = m+1, m+2, \dots.$$

Because

$$\langle h_j, \psi_i^m \rangle_{{}^{o}L^2_w} = \langle h_j(.), \mathbb{L}y R^m_y(.)|_{y=x_i} \rangle_{{}^{o}L^2_w} = \langle h_j(.), \sum_{k=2}^{m} h_k(.) \mathbb{L}y h_k(y)|_{y=x_i} \rangle_{{}^{o}L^2_w}$$

$$= \sum_{k=2}^{m} \langle h_j(.), h_k(.) \rangle_{{}^{o}L^2_w} \mathbb{L}y h_k(y)|_{y=x_i} = 0, \quad j = m+1, m+2, \dots.$$

Let

$$\Psi_m^{\perp} = \overline{Span\{h_i\}_{i=m+1}^{\infty}}.$$

So

$$u - u_m \in \Psi_m^{\perp},$$

and we have

$$||u - u_m||_{{}^{o}L^2_w} = || \sum_{i=m+1}^{\infty} \langle u - u_m, h_i \rangle_{{}^{o}L^2_w} h_i ||_{{}^{o}L^2_w}.$$

Thus

$$||u_m - u||_{{}^{o}L^2_w} \longrightarrow 0, \quad m \longrightarrow \infty.$$

In addition

$$||u - u_m||^2_{{}^{o}L^2_w} = || \sum_{i=m+1}^{\infty} \langle u - u_m, h_i \rangle_{{}^{o}L^2_w} h_i ||^2_{{}^{o}L^2_w}$$

$$= \sum_{i=m+1}^{\infty} (\langle u - u_m, h_i \rangle_{{}^{o}L^2_w})^2.$$

Clearly, $||u_m - u||_{{}^{o}L^2_w}$ is monotonically decreasing in m.

5 Numerical examples

In this section, we present several numerical experiments to evaluate the performance and accuracy of the proposed Chebyshev-based Reproducing Kernel Method (C-RKM). Comparisons are made with the classical RKM approach as reported in the literature. All numerical computations were carried out using Mathematica 10. The results demonstrate that C-RKM provides highly accurate solutions while requiring significantly fewer nodal points. This directly leads to reduced computational effort, especially in problems with singularities or sharp boundary/internal layers. Considering two-point boundary value problem

$$\begin{cases} p(x)u''(x) + \frac{1}{q(x)}u'(x) + \frac{1}{r(x)}u(x) = f(x), \quad x \in [a,b], \\ u(a) = \alpha, \quad u(b) = \beta, \end{cases} \tag{11}$$

where $p(x), q(x) \in \mathbb{C}(a,b)$ and $q(a)q(b) = 0$ or $r(a)r(b) = 0$, $r(x), f(x)$ are sufficiently regular given functions and α, β are finite constants. After homogenization of boundary conditions, Eq. (11) can be transformed into the Eq. (8).

Example 5.1. [2] In Eq. (11), taking $p(x) = 1,\ q(x) = xe^{\frac{1}{x}},\ r(x) = e^{\frac{1}{x}}, \alpha = \beta = 0,\ \ f(x) = \frac{4e^{3x}}{x}(-1+x^2+x^3+2xe^{\frac{1}{x}}(-1+2x+2x^2))$, the exact solution is $u(x) = 4x(1-x)e^{2x}$. This example is a singular boundary value problem on $[0,1]$. This class of problems arises very frequently in various fields of physics and engineering sciences, such as fluid mechanics, fluid dynamics, elasticity, reaction-diffusion process, chemical kinetics, and other branches of applied mathematics [2]. We compare numerical results of C-RKM and RKM [2] in Table 1 with $a = 0, b = 1, x_i = \cos(\frac{(2i+1)\pi}{2(m-1)}),\ i = 0, \ldots, m-2,\ m = 11$.

Example 5.2. [14] In Eq.(11), taking $p(x) = 1,\ q(x) = \sqrt{x},\ r(x) = x, \alpha = \beta = 0,\ \ f(x) = \frac{(2\sqrt{x}+x-4x\sqrt{x}-x^2)cos(x)-(-1+\sqrt{x}+2x+2x\sqrt{x}-x^2\sqrt{x})sin(x)}{\sqrt{x}}$, the exact solution is $u(x) = (x - x^2)sin(x)$. This example is another singularly boundary value problem on $[0,1]$. The numerical results of this problem are given in Table 2 with $a = 0, b = 1, x_i = \cos(\frac{(2i+1)\pi}{2(m-1)}),\ i = 0, \ldots, m-2,\ m = 6, 7, 8, 9$.

Example 5.3. [2] In Eq.(11), taking $p(x) = 2^{-10},\ q(x) = x,\ r(x) = \frac{1}{1+x^2}, \alpha = \beta = 0,\ f(x) = \frac{1}{512}xe^{x^2}(1537 + 514x^2) + e^x(-1 - \frac{1025x}{1024} - x^3)$, the exact solution is $u(x) = e^{x^2} - e^x$. This example is a singularly perturbed two point singular boundary value problem on $[0,1]$. A family of such problems are generally encountered in fluid and quantum mechanics , aerodynamics , reaction–diffusion process, optimal control, geophysics, chemical-reactor theory, etc. We present the numerical results of our C-RKM method and ordinary RKM (conventional method [2]) in Table 3 with $a = 0, b = 1, x_i = \cos(\frac{(2i+1)\pi}{2(m-1)}),\ i = 0, \ldots, m-2,\ m = 2, 3, \ldots, 10$. and Table 4, respectively.

Example 5.4. [18] Considering the following problem

$$\begin{cases} \varepsilon u''(x) + xu'(x) - u(x) = 0, \quad -1 < x < 1, \\ u(-1) = 1, \quad u(1) = 2. \end{cases} \tag{12}$$

This problem has an internal layer of width $O(\sqrt{\varepsilon})$. The exact solution is

$$u(x) = \frac{2\sqrt{\varepsilon}(x + 3e^{-\frac{x^2-1}{2\varepsilon}}) + e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}x\ erf(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}) + 3e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}x\ erf(\frac{x}{\sqrt{2}\sqrt{\varepsilon}})}{2e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\ erf(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}) + 4\sqrt{\varepsilon}}.$$

Node	Exact	RKM	C-RKM	Absolute error	
	$u(x)$	$u_{200}(x)$[2]	$u_{10}(x)$	RKM	C-RKM
$\frac{1}{200}$	0.0201000	0.0201008	0.0201000	8.0E-07	0.0
$\frac{17}{200}$	0.3687483	0.368761	0.368748	1.27E-05	3.0E-07
$\frac{33}{200}$	0.7665625	0.766585	0.766562	2.25E-05	5.0E-07
$\frac{49}{200}$	1.207751	1.20778	1.20775	2.9E-05	1.0E-06
$\frac{65}{200}$	1.680887	1.68092	1.68089	3.3E-05	3.0E-06
$\frac{81}{200}$	2.166759	2.16679	2.16676	3.1E-05	1.0E-06
$\frac{97}{200}$	2.635570	2.6356	2.63557	3.0E-05	0.0
$\frac{113}{200}$	3.043340	3.04337	3.04334	3.0E-05	0.0
$\frac{129}{200}$	3.327269	3.32729	3.32727	2.1E-05	1.0E-06
$\frac{145}{200}$	3.399834	3.39985	3.39983	1.6E-05	4.0E-06
$\frac{161}{200}$	3.141265	3.14127	3.14127	5.0E-06	5.0E-06
$\frac{177}{200}$	2.390024	2.39002	2.39002	4.0E-06	4.0E-06
$\frac{193}{200}$	0.9307728	0.930762	0.930773	1.08E-05	2.0E-07

Table 1: Numerical results for Example 5.1 .

These problems arise frequently in other field of science such as oceanic and atmospheric circulation, chemical reactions, geophysical fluid dynamics, optimal control, etc. In these problems, there exists a small parameter that multiplies the highest order derivative and a boundary or interior layer where the solutions change rapidly [18]. The numerical results are given in Table 5 with $x_i = \cos(\frac{(2i+1)\pi}{2(m-1)}),\ i = 0, \ldots, m-2,\ m = 13$.

Table 1 indicate that C-RKM is able to solve this example with only 10 nodal points (only 10 times the Gram-Schmidt process). C-RKM also achieves much higher precision (up to 100 times higher in some nodes). Note that, the precision of the conventional method(RKM) has been obtained by applying 200 nodal points. It means that by using 200 times the Gram-Schmidt process has been achieved. applying only 10 times the Gram-Schmidt process in C-RKM instead of 200 times in RKM shows that, the required computational time to achieve the desired accuracy reduced significantly. The absolute error of C-RKM for $u_{10}(x)$ is shown in Fig. 1. Table 2 shows that the our approximate solutions, with only 8 nodal points are more accurate than the results of the [14] with 103 nodal points. Also, it can easily be seen that an increment in m from 5 to 8 can improve the accuracy of the solutions up to 10^3 times. Such speed of convergence cannot be seen in [14]. This suggests that sole dependence of convergence of C-RKM on number of sample points is dropped. The absolute error of C-RKM with different values of m is shown in Fig. 2. Rapid decline absolute error by increasing m can easily be seen in this figure.

Numerical results presented in Table 3 and Fig. 3 show that, firstly, RMS error decreases as the number of points increases. Secondly, with the addition of just one nodal point, the error is

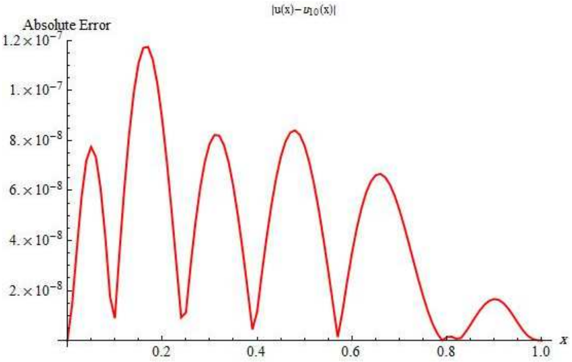

Figure 1: Absolute error of $u_{10}(x)$ for the Example 1.

Node	$m = 6$	$m = 7$	$m = 8$	$m = 9$	$u_{103}(x)$[14]
0.001	1.15768E-8	2.99683E-11	2.49249E-10	7.45715E-12	7.9E-13
0.08	6.51033E-7	8.11139E-9	3.31238E-7	2.10122E-9	5.1E-07
0.16	1.56331E-6	7.09387E-8	2.60434E-7	8.21754E-11	2.1E-06
0.32	6.61056E-6	3.90951E-8	3.88679E-7	1.11810E-9	7.8E-06
0.48	5.41480E-7	1.27819E-7	2.16129E-7	1.144810E-9	1.4E-05
0.64	8.11503E-6	4.77257E-8	5.03970E-7	3.03362E-10	1.7E-05
0.80	2.59482E-6	1.24804E-7	1.04487E-7	3.91390E-9	1.4E-05
0.96	7.98685E-7	1.82312E-8	4.12686E-8	7.47708E-10	3.9E-06
1.00	0	0	0	0	0
CPU Time(s)	1.778	3.915	8.268	18.097	

Table 2: Absolute Error $|u(x) - u_{m-1}(x)|$ and CPU Time for Example 5.2 .

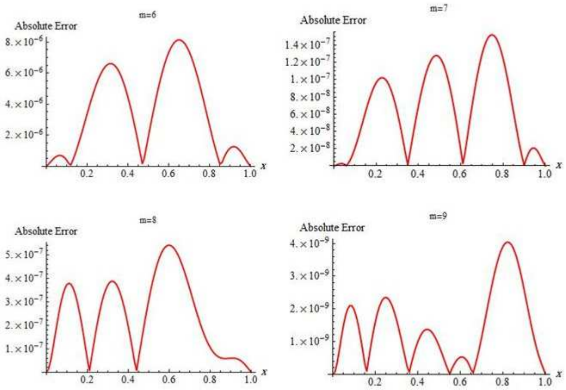

Figure 2: Absolute error for the Example 2, with different values of m.

m	Number of x_i	RMS Error C-RKM	CPU Time (s)
2	1	5.87745E-1	0.0
3	2	4.18245E-2	0.047
4	3	2.77845E-2	0.171
5	4	1.63021E-3	0.671
6	5	5.34107E-4	1.42
7	6	2.70943E-5	3.822
8	7	8.51409E-6	6.927
9	8	3.5382E-7	14.758
10	9	1.07207E-7	25.678

Table 3: Root Mean Square Error for Example 5.3 .

Number of x_i	RMS Error RKM old method [2]
8	3.18851E-03
16	3.7987E-04
32	4.10187E-05
64	4.02599E-06
128	3.47918E-07

Table 4: Root Mean Square Error for Example 5.3 by RKM (old method) .

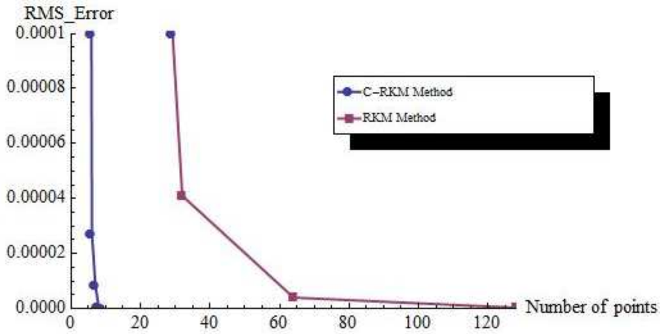

Figure 3: The rate of reduce RMS error in terms of increasing the number of nodal points in RKM and C-RKM.

Node	Absolute Error
-0.9	1.11022E-16
-0.7	1.11022E-16
-0.5	1.66533E-16
-0.3	3.46857E-4
-0.1	1.04485E-5
0	3.57588E-3
0.1	1.04484E-5
0.3	3.46857E-4
0.5	2.22045E-16
0.7	2.22045E-16
0.9	2.22045E-16

Table 5: $|u(x) - u_{12}(x)|$ for Example 5.4 .

reduced 10 times. However, the results in Table 4 show that, in the RKM, decreasing this rate of error (10 times) depends on doubling the number of nodal points. Also, comparing the results of these two tables shows that the error in our method(C-RKM) with 9 nodal points is about three times smaller than that of the ordinary method with 128 nodal points. This strongly suggests that, again, the sole dependence of convergence on the number of nodal points is dropped in C-RKM. Numerical solutions presented in Table 5 are obtained from C-RKM with only 12 nodal points. Our results are acceptable compared to the results of [18] with 200 and 400 points.

6 Conclusions

In this study, we developed a Chebyshev-based reproducing kernel method (C-RKM) for efficiently solving two-point boundary value problems. The proposed approach retains the essential theoretical strengths of classical reproducing kernel methods while addressing two common limitations: the heavy dependence on dense nodal distributions and the computational cost associated with repeated orthogonalization. By constructing the reproducing kernel using shifted Chebyshev polynomials that satisfy boundary conditions intrinsically, we established a dynamic solution space that evolves with the polynomial degree m. This flexibility allows for accurate approximations using relatively few nodal points, without compromising convergence or stability. Numerical experiments confirmed the theoretical advantages of C-RKM. Compared to conventional RKM techniques, our method consistently required fewer nodes and yielded significantly better accuracy, particularly in problems featuring singularities or internal layers. The rapid error decay and low computational overhead make C-RKM a promising tool for practical applications in applied mathematics and engineering. Future work may explore extending this framework to higher-order boundary value problems, systems of equations, or multi-dimensional domains using tensor-product Chebyshev kernels.

Acknowledgements

We thank the anonymous reviewers for helpful comments, which lead to definite improvement in the manuscript.

References

[1] Aronszajn, N., (1950). Theory of reproducing kernels. *Trans. Amer. Math. Soc.*, 68(3), 337–404. doi:10.1090/S0002-9947-1950-0051437-7

[2] Cui, M., & Lin, Y. (2009). *Nonlinear Numerical Analysis in Reproducing Kernel Space.* Nova Science Publisher, New York.

[3] Abbasbandy, S., & Azarnavid, B., & Alhuthali, M.S. (2015). A shooting reproducing kernel Hilbert space method for multiple solutions of nonlinear boundary value problems. *Journal of Computational and Applied Mathematics*, 279, 293–305. doi:10.1016/j.cam.2014.11.014

[4] Geng, F., & Li, X.M. (2012). A new method for Riccati differential equations based on reproducing kernel and quasi-linearization methods. *Abs. Appl. Anal.*, 2012, 603748. doi:10.1155/2012/603748

[5] Mohammadi, M., & Mokhtari, R. (2011). Solving the generalized regularized long wave equation on the basis of a reproducing kernel space. *Journal of Computational and Applied Mathematics*, 235(14), 4003–4014. doi:10.1016/j.cam.2011.02.012

[6] Mokhtari, R., & Toutian Isfahani, F., & Mohammadi, M. (2012). Reproducing Kernel Method for Solving Nonlinear Differential-Difference Equations. *Abs. Appl. Anal.*, 2012, 514103. doi:10.1155/2012/514103

[7] Schaback, R., (2007). Convergence of unsymmetric kernel-based meshless collocation methods. *SIAM J. Numer. Anal.*, 45(1), 333–351. doi:10.1137/05063336

[8] Mohammadi, M., & Mokhtari, R., & Panahipour, H. (2013). A Galerkin-reproducing kernel method: application to the 2D nonlinear coupled Burgers' equations. *Eng. Anal. Bound. Elem.*, 37(12), 1642–1652. doi:10.1016/j.enganabound.2013.09.005

[9] Cialenco, I., & Fasshauer, G.E. & Ye, Q. (2012). Approximation of stochastic partial differential equations by a kernel-based collocation method. *Int. J. Comput. Math.*, 89(18), 2543–2561. doi:10.1080/00207160.2012.688111

[10] Ketabchi, R., & Mokhtari, R., & Babolian, E. (2015). Some error estimates for solving Volterra integral equations by using the reproducing kernel method. *J. Comput. Appl. Math.*, 273, 245–250. doi:10.1016/j.cam.2014.06.016

[11] Sedaghat, S., & Ordokhani, Y., & Dehghan, M. (2012) Numerical solution of the delay differential equations of pantograph type via Chebyshev polynomials. *Commun. Nonlinear Sci. Numer. Simulat.*, 17(12), 4815–4830. doi:10.1016/j.cnsns.2012.05.009

[12] Du, H., & Cui, M.G. (2008). Approximate solution of the Fredholm integral equation of the first kind in a reproducing kernel Hilbert space. *Appl. Math. Lett.*, 21(6), 617–623. doi:10.1016/j.aml.2007.07.014

[13] Fasshauer, G.E. & Hickernell, F.J., & Ye, Q. (2015). Solving support vector machines in reproducing kernel Banach spaces with positive definite functions. *Appl. Comput. Harmon. Anal.*, 38(1), 115–139. doi:10.1016/j.acha.2014.03.007

[14] Cui, M.G., & Geng, F.Z. (2007). Solving singular two-point boundary value problem in reproducing kernel space. *J. Comput. Appl. Math.*, 205(1), 6–15. doi:10.1016/j.cam.2006.04.037

[15] Geng, F.Z., & Cui, M.G., & Zhang, B. (2010). Method for solving nonlinear initial value problems by combining homotopy perturbation and reproducing kernel Hilbert space methods. *Nonlinear Analysis: Real World Applications*, 11(2), 637–644. doi:10.1016/j.nonrwa.2008.10.033

[16] Geng, F.Z., & Cui, M.G. (2007). Solving a nonlinear system of second order boundary value problems. *J. Math. Anal. Appl.*, 327(2), 1167–1181. doi:10.1016/j.jmaa.2006.05.011

[17] Geng, F.Z., & Cui, M.G. (2008). Solving singular nonlinear two-point boundary value problems in the reproducing kernel space. *J. Korean Math. Soc.*, 45, 631–644. doi:10.1016/j.cam.2006.04.037

[18] Geng, F.Z., & Qian, S.P., & Li, S. (2014). A numerical method for singularly perturbed turning point problems with an interior layer. *J. Comput. Appl. Math.*, 255, 97–105. doi:10.1016/j.cam.2013.04.040

[19] Geng, F.Z., & Cui, M.G. (2012). A reproducing kernel method for solving nonlocal fractional boundary value problems. *Appl. Math. Lett.*, 25(5), 818–823. doi:10.1016/j.aml.2011.10.025

[20] Niu, J., & Lin, Y.Z., & Zhang, C.P. (2012). Approximate solution of nonlinear multi-point boundary value problem on the half-line. *Modelling Anal.*, 17(2), 190–202. doi:10.3846/13926292.2012.660889

[21] Boyd, J.P. (2001). *Chebyshev and Fourier spectral methods, Second edition.* Dover Publishers, New York.

[22] Simmons, G.F. (1983). *Introduction to topology and modern analysis.* McGraw-Hill Book Company, Inc., New York.

[23] Cui, M.G., & Geng, F.Z. (2007). An efficient computational method for second order boundary value problems of nonlinear differential equations. *Appl. Math. Comput.*, 194(2), 354–365. doi:10.1016/j.amc.2007.04.029

[24] Cui, M.G., & Geng, F.Z. (2007). Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space. *Appl. Math. Comput.*, 192(2), 389–398. doi:10.1016/j.amc.2007.03.016

[25] Geng, F.Z. (2009). Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method. *Appl. Math. Comput.*, 215(6), 2095–2102. doi:10.1016/j.amc.2009.08.002

[26] Cui, M.G., & Geng, F.Z. (2007). A computational method for solving one-dimensional variable-coefficient Burgers equation. *Appl. Math. Comput.*, 188(2), 1389–1401. doi:10.1016/j.amc.2006.11.005

[27] Cui, M.G., & Chen, Z. (2007). The exact solution of nonlinear age-structured population model. *Nonlinear Analysis: Real World Applications*, 8(4), 1096–1112. doi:10.1016/j.nonrwa.2006.06.004

[28] Li, C.L., & Cui, M.G (2003). The exact solution for solving a class of nonlinear operator equations in the reproducing kernel space. *Appl. Math. Comput.*, 143(2-3), 393–399. doi:/10.1016/S0096-3003(02)00370-3

[29] Xie, S.S., & Heo, S., & Kim, S., & Woo, G., & Yi, S. (2008). Numerical solution of one-dimensional Burgers equation using reproducing kernel function. *J. Comput. Appl. Math.*, 214(2), 417–434. doi:/10.1016/j.cam.2007.03.010

[30] Yao, H., & Lin, Y. (2011). New algorithm for solving a nonlinear hyperbolic telegraph equation with an integral condition. *Internat. J. Numer. Methods. Bio. Engrg.*, 27(10), 1558–1568. doi:10.1002/cnm.1376

[31] Zhou, Y., & Cui, M.G., & Lin, Y. (2009). Numerical algorithm for parabolic problems with non-classical conditions. *J. Comput. Appl. Math.*, 230(2), 770–780. doi:10.1016/j.cam.2009.01.012

[32] Zhou, Y., & Cui, M.G., & Lin, Y. (2008). The solution of a parabolic differential equation with non-local boundary conditions in the reproducing kernel space. *Appl. Math. Comput.*, 202(2), 708–714. doi:10.1016/j.amc.2008.03.008

[33] Du, H., & Shen, J. (2008). Reproducing kernel method of solving singular integral equation with cosecant kernel. *J. Math. Anal. Appl.*, 348(1), 308–314. doi:10.1016/j.jmaa.2008.07.037

[34] Yang, L., & Cui, M.G. (2006). New algorithm for a class of nonlinear integro-differential equations in the reproducing kernel space. *Appl. Math. Comput.*, 174(2), 942–960. doi:10.1016/j.amc.2005.05.026

[35] Deutsch, F. (2001). *Best approximation in inner product spaces.* Springer-Verlag, New York, Inc.

[36] Niu, J., & Lin, Y., & Cui, M.G. (2013). A novel approach to calculation of reproducing kernel on infinite interval and applications to boundary value problems. *Abstract and Applied Analysis*, 2013, 959346. doi:10.1155/2013/959346

[37] Niu, J., & Li, P. (2014). Numerical algorithm for the third-order partial differential equation with three-point boundary value problem. *Abstract and Applied Analysis*, 2014, 630671. doi:10.1155/2014/630671

[38] Niu, J., & Lin, Y.Z., & Zhang, C.P. (2012). Numerical solution of nonlinear three-point boundary value problem on the positive half-line. *Mathematical Methods in the Applied Sciences*, 35(13), 1601–1610. doi:10.1002/mma.2549

[39] Niu, J., & Lin, Y.Z., & Cui, M.G. (2012). Approximate solutions to three-point boundary value problems with two-space integral condition for parabolic equations. *Abstract and Applied Analysis*, 2012, 414612. doi:10.1155/2012/414612

[40] Lin, Y.Z., & Niu, J., & Cui, M.G. (2012). A numerical solution to nonlinear second order three-point boundary value problems in the reproducing kernel space. *Applied Mathematics and Computation*, 218(14), 7362–7368. doi:10.1016/j.amc.2011.11.009